Math 846

Lecture 5

Prop 18 Fix a vertex x of $H(D,2)$ and write $A^* = A^*(x)$. Then

(i) $A^2A^* - 2AA^*A + A^*A^2 = 4A^*$;

(ii) $(A^*)^2A - 2A^*AA^* + A(A^*)^2 = 4A$.

pf (i) Pick $y, z \in X$.

Define

$r =$ (y,z)-entry of A^2A^*

$s =$ " AA^*A

$t =$ " A^*A^2

$u =$ " A^*

Show $r - 2s + t = 4u$.

We consider some cases.

Write $i = \partial(x,y)$.

Case	r	s	t	u
$y = z$	$D\theta_i^*$	$i\theta_{i-1}^* + (D-i)\theta_{i+1}^*$	$D\theta_i^*$	θ_i^*
$\partial(y,z) = 2$ and $\partial(x,z) = i-2$	$2\theta_{i-2}^*$	$2\theta_{i-1}^*$	$2\theta_i^*$	0
$\partial(y,z) = 2$ and $\partial(x,z) = i$	$2\theta_i^*$	$\theta_{i-1}^* + \theta_{i+1}^*$	$2\theta_i^*$	0
$\partial(y,z) = 2$ and $\partial(x,z) = i+2$	$2\theta_{i+2}^*$	$2\theta_{i+1}^*$	$2\theta_i^*$	0
none of the above	0	0	0	0

In each case

$$r - 2s + t = 4u$$

✓

(ii) Pick $y, z \in X$.

Define

$r =$ (y,z)-entry of $(A^*)^2 A$

$s =$ " " $A^* A A^*$

$t =$ " " $A (A^*)^2$

$u =$ " " A

show $r - 2s + t = 4u$.

Consider cases. Let $i = \partial(x,y)$

Case	r	s	t	u
$\partial(y,z)=1$ and $\partial(x,z)=i-1$	θ_i^{*2}	$\theta_i^* \theta_{i-1}^*$	θ_{i-1}^{*2}	1
$\partial(y,z)=1$ and $\partial(x,z)=i+1$	θ_i^{*2}	$\theta_i^* \theta_{i+1}^*$	θ_{i+1}^{*2}	1
none of the above	0	0	0	0

In each case

$r - 2s + t = 4u$ ✓

□

Recall that $H(D,2)$ has distinct eigenvalues

$$\theta_i = D - 2i \qquad 0 \le i \le D$$

For $0 \le i \le D$ let

$$V_i = \text{eigenspace for } A \text{ with eigenvalue } \theta_i$$

Prop 19 For a vertex x of $H(D,2)$
the matrix $A^* = A^*(x)$ satisfies

$$E_i A^* E_j = 0 \text{ if } |i-j| \ne 1 \qquad (0 \le i,j \le D)$$

Moreover for $0 \le i \le D$,

$$A^* V_i \subseteq V_{i-1} + V_{i+1}$$

where $\quad V_{-1} = 0, \quad V_{D+1} = 0$

pf For $0 \le i, j \le D$ with $|i-j| \neq 1$

show $E_i A^* E_j = 0$

By Prop 18 (i),

$$0 = A^2 A^* - 2AA^*A + A^*A^2 - 4A^*$$

So

$$0 = E_i \left(A^2A^* - 2AA^*A + A^*A^2 - 4A^* \right) E_j$$

$$= E_i A^* E_j \left(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4 \right)$$

$$= E_i A^* E_j \left(\theta_i - \theta_j - 2 \right)\left(\theta_i - \theta_j + 2 \right)$$

$$\theta_i - \theta_j - 2 = D - 2i - (D - 2j) - 2$$

$$= 2(j - i - 1)$$

$$\neq 0$$

$$\theta_i - \theta_j + 2 = 2(j - i + 1)$$

$$\neq 0$$

So $0 = E_i A^* E_j$.

For $0 \le i \le D$ show

$$A^* V_i \subseteq V_{i-1} + V_{i+1}$$

$$
\begin{aligned}
A^* V_i &= A^* E_i V \\
&= I A^* E_i V \\
&= \left(\sum_{l=0}^{D} E_l \right) A^* E_i V \\
&= \underbrace{E_{i-1} A^* V}_{\subseteq E_{i-1} V = V_{i-1}} + \underbrace{E_{i+1} A^* V}_{\subseteq E_{i+1} V = V_{i+1}} \\
&\subseteq V_{i-1} + V_{i+1}
\end{aligned}
$$

$\square$

To summarize for $H(D,2)$,

- $A^* V_i \subseteq V_{i-1} + V_{i+1}$ $\quad 0 \le i \le D$
- $A V_i^* \subseteq V_{i-1}^* + V_{i+1}^*$ $\quad 0 \le i \le D$

  $V_i^* = E_i^* V$

- $\{V_i\}_{i=0}^D$ is an ordering of the eigenspaces of A
- $\{V_i^*\}_{i=0}^D$ " " " A^*

Aside on linear algebra

Until further notice let $\mathbb{F}$ = any field whatsoever

DEF 20 Let V denote a non 0, finite-dimensional vector space over $\mathbb{F}$.

A <u>tridiagonal pair on V</u> is an ordered pair of diagonalizabe $\mathbb{F}$-linear maps

$$A : V \to V \quad \text{and} \quad A^* : V \to V$$

such that

(i) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \le i \le d$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(ii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad 0 \le i \le \delta$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iii) there does not exist a subspace $W \subseteq V$ st

$$AW \subseteq W, \quad A^*W \subseteq W, \quad W \neq 0, \quad W \neq V$$

Note For the tridiagonal pair A, A^* in DEF 20, it is a theorem that $d = \delta$, but we dont assume $d = \delta$ in advance.

DEF 21 For the tridiagonal pair A, A^* in DEF 20,

(i) A, A^* is called <u>bipartite</u> whenever

$$A V_i^* \subseteq V_{i-1}^* + V_{i+1}^* \qquad 0 \le i \le \delta$$

(ii) A, A^* is called <u>dual bipartite</u> whenever

$$A^* V_i \subseteq V_{i-1} + V_{i+1} \qquad 0 \le i \le d.$$

Back to our graph $H(D,2)$ with adjacency matrix A and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Fix a vertex x of $H(D,2)$ and write $A^* = A^*(x)$, $T = T(x)$.

Let W denote an irreducible T-module.

Then the pair A, A^* acts on W as a tridiagonal pair that is bipartite and dual bipartite (ex).

Next goal: relate $H(D,2)$ to the Lie algebra sl_2.

Until further notice $\mathbb{F} = \mathbb{C}$. Recall

$$i \in \mathbb{C} \qquad i^2 = -1$$

Fix a vertex x of $H(D,2)$ and write

$A^* = A^*(x)$, $\quad T = T(x)$

Define matrices B, C by

$$B = A^*, \qquad [A, B] = 2iC$$

where

$$[\alpha, \beta] = \alpha\beta - \beta\alpha$$

LEM 22 With above notation,

(i) $[A, B] = 2iC$,

(ii) $[B, C] = 2iA$,

(iii) $[C, A] = 2iB$.

pf (i) ✓

(ii), (iii) This is Prop 18 in disguise. $\square$

The Lie algebra $sl_2(\mathbb{C})$ consists of the 2×2 matrices over $\mathbb{C}$ that have trace 0, together with the Lie bracket

$$[\alpha, \beta] = \alpha\beta - \beta\alpha$$

$sl_2(\mathbb{C})$ has a basis

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad c = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

"Dirac basis"

and,
$$[a, b] = 2ic,$$
$$[b, c] = 2ia,$$
$$[c, a] = 2ib.$$

Thus for $H(0,2)$ the standard module V becomes an $sl_2(\mathbb{C})$-module on which a, b, c act as A, B, C respectively.

<u>Note</u> Another basis for $sl_2(\mathbb{C})$ is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$[h, e] = 2e,$$
$$[h, f] = -2f,$$
$$[e, f] = h.$$