
Combinatorics 475

Lecture 4

2.5 Combinations of Multisets

Given a multiset M

For $0 \leq r \leq |M|$

an r -combination of M is an unordered selection of r elements of M

ex

$$M = \{a, a, b, c\}$$

r	r -combinations of M
4	$\{a, a, b, c\}$
3	$\{a, a, b\}$ $\{a, a, c\}$, $\{a, b, c\}$
2	$\{a, a\}$ $\{a, b\}$ $\{a, c\}$ $\{b, c\}$
1	$\{a\}$ $\{b\}$ $\{c\}$
0	$\emptyset$

Given multiset

$$M = \{ n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k \}$$

Given $0 \leq r \leq |M|$

How many r -combinations of M ?

List them:

The r -combinations of M are

$$\{ x_1 \cdot a_1, x_2 \cdot a_2, \dots, x_k \cdot a_k \}$$

$$x_1, x_2, \dots, x_k \in \mathbb{Z}$$

$$0 \leq x_1 \leq n_1$$

$$0 \leq x_2 \leq n_2$$

$$\vdots$$

$$0 \leq x_k \leq n_k$$

$$x_1 + x_2 + \dots + x_k = r$$

} *

The number of r -combinations of M = # solutions to (*).

[no simple general formula We will
find general method in ch 6]

Special Case

Ref to (*), the condition

$$x_1 + x_2 + \dots + x_k = r$$

implies

$$x_1 \leq r$$

$$x_2 \leq r$$

⋮

$$x_k \leq r$$

So if each of $n_1, n_2, \dots, n_k$ is at least r then the upper bounds

$$x_1 \leq n_1$$

⋮

$$x_k \leq n_k$$

play no role. In this case the problem * becomes:

Ex. Given integers $k \geq 1$ and $r \geq 0$

Find the number of integer solutions to

$$x_1 \geq 0, x_2 \geq 0, \dots, x_k \geq 0$$

} **

$$x_1 + x_2 + \dots + x_k = r$$

take $r = 2$ and $k = 3$ Integral
solutions:

x_1	x_2	x_3
2	0	0
0	2	0
0	0	2
1	1	0
1	0	1
0	1	1

integral sols = 6

Sol Given an integer sol to $**$:

$$x_1, x_2, \dots, x_k$$

Consider r dots in a row

Partition the set of dots into groups of size

$$x_1, x_2, \dots, x_k$$

Separate adjacent groups with a marker $|$

Note $x_1, x_2, \dots, x_k$ are determined by the location of the markers

$$\left[\begin{array}{cccc} r=8 & k=4 & & \\ & x_1=3 & x_2=1 & x_3=0 & x_4=4 \\ & \cdot \cdot \cdot | \cdot | | \cdot \cdot \cdot & & \end{array} \right]$$

Symbol	# used
$\cdot$	r
$ $	$k-1$

$$\# \text{ symbols used} = r+k-1$$

$$\# \text{ integer sols to } ** = \# \text{ ways to fill } r+k-1 \text{ blanks with } r \text{ dots and } k-1 \text{ 's}$$

$$= \# \text{ } r\text{-subsets of } r+k-1 \text{ element set}$$

$$= \# (k-1)\text{-subsets of } r+k-1 \dots$$

$$= \binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

□

In summary

then Given a multiset

$$M = \{ n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k \}$$

Given $0 \leq r \leq |M|$

The number of r -combinations of M is

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

provided each of $n_1, n_2, \dots, n_k$ is at least r .

□

Ex Find the number of integer solutions to

$$x_1 \geq 3, \quad x_2 \geq 0, \quad x_3 \geq 1$$

$$x_1 + x_2 + x_3 = 11$$

Sol. Change variables: *define*

$$y_1 = x_1 - 3$$

$$y_2 = x_2$$

$$y_3 = x_3 - 1$$

Requirement becomes

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0$$

$$y_1 + y_2 + y_3 = 7 \quad (= 11 - 3 - 1)$$

$$\text{ans} = \binom{7 + 3 - 1}{3 - 1} = \binom{9}{2} = 36$$

□

2.6 Probability

Recall game of Poker

View 52 card deck as Cartesian product

13 ranks

	2	3	4	5	6	7	8	9	10	J ⁼¹¹	Q ⁼¹²	K ⁼¹³	A ⁼¹⁴ "1"
4 suits													
C	.	.	.										
D	.	.	.										
H													
S													

Poker hand is 5 cards ie 5-element subset of Deck

Each Poker hand equally likely

Ex In a poker hand, what is the probability
of getting a "full house"?

3 cards of one rank

2 cards of another rank

Sol "Sample space" S is set of all possible
poker hands

Each element of S equally likely

$$|S| = \binom{52}{5}$$

Lec 4 8

"Event" E is the set of full-house poker hands.

$$\text{Prob}(E) = \frac{|E|}{|S|}$$

Find $|E|$.

We construct a full-house in stages:

stage	to do	# choices
1	pick the 3-card rank	13
2	pick the suits for prev 3 cards	$\binom{4}{3} = 4$
3	pick the 2-card rank	12
4	pick the suits for prev 2 cards	$\binom{4}{2}$

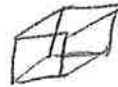
$$|E| = 13 \times 4 \times 12 \times \binom{4}{2}$$

$$\text{Prob}(E) = \frac{13 \times 4 \times 12 \times \binom{4}{2}}{\binom{52}{5}}$$

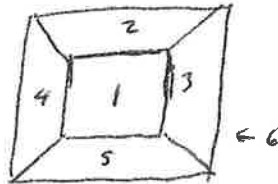
□

Recall game of dice

one die is a cube



faces are numbered



For a fair die each number equally likely to occur on top

Ex 3 dice are rolled.

What is probability that the sum of the top numbers is 8?

Sol Sample space S consists of all 3-tuples

$$(x, y, z)$$

$$1 \leq x, y, z \leq 6$$

$$|S| = 6^3$$

Each outcome in S equally likely

Event E consists of the 3-tuples (x, y, z) in S such that

$$x + y + z = 8$$

$|E| = \#$ integer sol to

$$1 \leq x \leq 6, \quad 1 \leq y \leq 6, \quad 1 \leq z \leq 6$$

$$x + y + z = 8$$

ch vars

$$X = x - 1$$

$$Y = y - 1$$

$$Z = z - 1$$

= # integer solutions

$$0 \leq X \leq 5, \quad 0 \leq Y \leq 5, \quad 0 \leq Z \leq 5$$

$$X + Y + Z = 5$$

= # integer solutions

$$0 \leq X, \quad 0 \leq Y, \quad 0 \leq Z$$

$$X + Y + Z = 5$$

$$= \binom{5+3-1}{3-1} = \binom{7}{2} = 21$$

So

$$\text{Prob}(E) = \frac{|E|}{|S|}$$

$$= \frac{21}{6^3}$$

□

Ex In a Poker hand, what is probability
of getting "two pairs" :

2 cards of one rank
2 cards of another rank
1 card of 3rd rank

Sol Sample space S consists of all poker hands

Event E is set of all "two pair" poker hands

Find $|E|$

Construct "two pair" hand in stages :

stage	to do	# choices
1	pick the ranks of the 2 pairs	$\binom{13}{2}$
2	pick suits for the pair with low rank	$\binom{4}{2}$
3	... high rank	$\binom{4}{2}$
4	pick 5th card	$52 - 8 = 44$

$|E|$ is product of $\uparrow$

$$\text{Prob}(E) = \frac{|E|}{|S|} = \frac{\binom{13}{2} \binom{4}{2}^2 44}{\binom{52}{5}}$$

□