Math 846

Lecture 4

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

We continue to discuss a connected graph $\Gamma = (X, \mathcal{R})$. Fix $x \in X$ and write

$M^* = M^*(x)$, $\quad T = T(x)$.

We seek a matrix $A^* = A^*(x)$ that generates M^* and

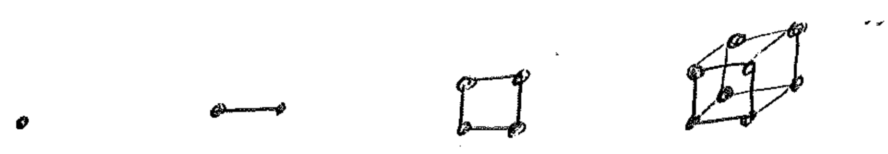
$$E_i A^* E_j = 0 \quad \text{if } |i-j| > 1 \qquad (0 \le i,j \le r)$$

A^* might not exist. The existence of A^* gives T extra structure that we will explore.

We begin this exploration with an extended example, the hypercubes $H(D,2)$

D	0	1	2	3	$\cdots$

$H(D,2)$

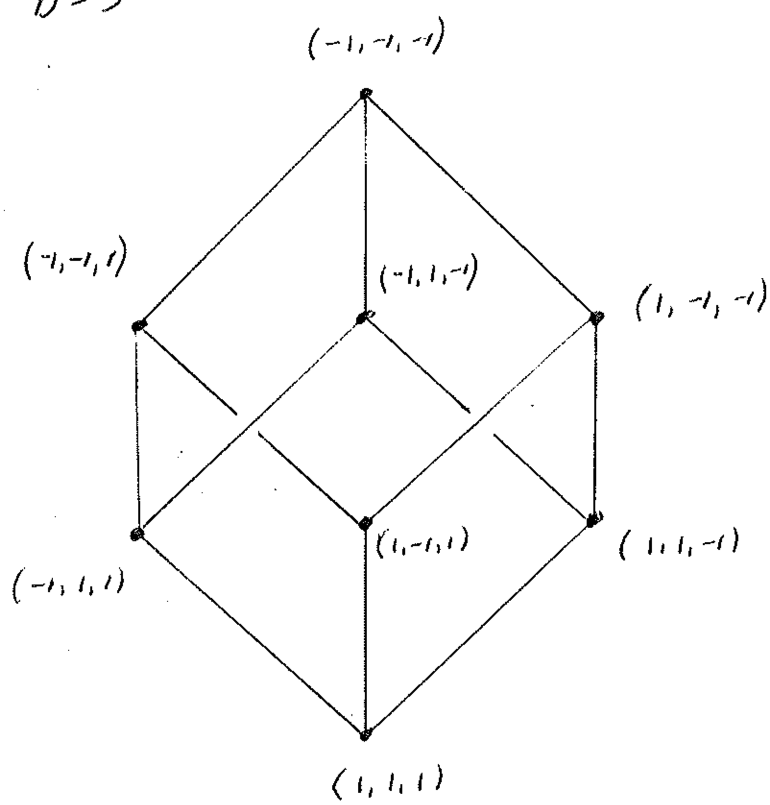


DEF 11 For $D \in \mathbb{N}$ the <u>hypercube</u> $H(D,2)$ has vertex set X consisting of the D-tuples

$$(a_1, a_2, \dots, a_D) \qquad a_i \in \{1, -1\} \qquad 1 \le i \le D$$

Vertices $x, y \in X$ are adjacent whenever they differ in exactly one coordinate.

Ex $D=3$



Observations about $H(D,2)$:

- $H(D,2)$ is connected
- $H(D,2)$ is regular with valency D
- $H(D,2)$ is bipartite
- $d(x) = D \qquad \forall x \in X$
- $H(D,2)$ has diameter D
- For $x, y \in X$,

  $\partial(x,y) =$ # coordinates at which x, y differ

Next goal: Find the eigenvalues of $H(D,2)$

We will use the fact that $H(D,2)$ is a Cartesian product, as we now explain.

DEF 12 For graphs

$$\Gamma = (X, \mathcal{R}), \qquad \Gamma' = (X', \mathcal{R}')$$

their <u>Cartesian product</u> is the graph $\Gamma \times \Gamma'$

with vertex set

$$X \times X' = \{ (a,b) \mid a \in X, \ b \in X' \}$$

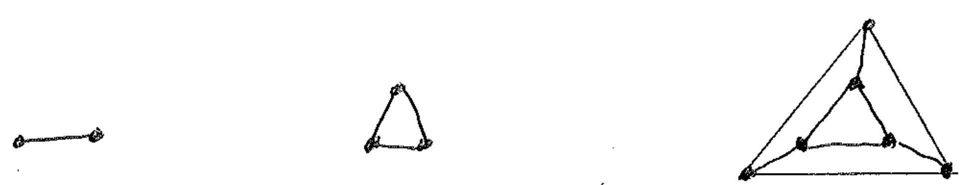
Vertices (a,b) and (c,d) are adjacent in $\Gamma \times \Gamma'$ whenever

- $a = c$ and b, d are adjacent in Γ'

or

- a, c are adjacent in Γ and $b = d$

EX 13 Take $\Gamma = K_2$, $\Gamma' = K_3$



Γ $\qquad$ Γ' $\qquad$ $\Gamma \times \Gamma'$

LEM 14 Given a graph $\Gamma = (X, R)$ with eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n \qquad n = |X|$$

Given a graph $\Gamma' = (X', R')$ with eigenvalues

$$\mu_1, \mu_2, \ldots, \mu_m \qquad m = |X'|$$

Then the graph $\Gamma \times \Gamma'$ has eigenvalues

$$\lambda_i + \mu_j \qquad (1 \le i \le n, \quad 1 \le j \le m)$$

pf Identify the standard module of $\Gamma \times \Gamma'$ with

$$V \otimes V' \qquad \otimes = \otimes_{\mathbb{F}}$$

where V = standard module for Γ
V' = " Γ'

For a vertex z of $\Gamma \times \Gamma'$ write $z = (a, b) \in X \times X'$

View $\hat{z} = \hat{a} \otimes \hat{b} \in V \otimes V'$

Adjacency matrix of $\Gamma \times \Gamma'$ becomes

$$\underset{\substack{\uparrow \\ \text{adj matrix of } \Gamma}}{A} \otimes I' + I \otimes \underset{\substack{\uparrow \\ \text{adj matrix of } \Gamma'}}{A'}$$

If $u \in V$ is an eigenvector for A with eigenvalue λ
$v \in V'$ is an eigenvector for A' with eigenvalue μ

Then
$u \otimes v$ is an eigenvector for $A \otimes I' + I \otimes A'$
with eigenvalue $\lambda + \mu$.

The result follows. $\square$

LEM 15 For $D \in \mathbb{N}$ the hypercube

$H(D,2)$ has spectrum

$$\begin{pmatrix} \theta_0, \theta_1, \ldots, \theta_D \\ m_0\ m_1\ \ldots\ m_D \end{pmatrix}$$

with

$$\theta_i = D - 2i \qquad (0 \le i \le D)$$

$$m_i = \binom{D}{i}.$$

pf $H(D,2)$ is the Cartesian product of D copies of the complete graph K_2:

$$H(D,2) = \underbrace{K_2 \times K_2 \times \cdots \times K_2}_{D}$$

The eigenvalues of K_2 are $1, -1$.

By LEM 14 the eigenvalues of $H(D,2)$ are the sums

$$\lambda_1 + \lambda_2 + \cdots + \lambda_D \qquad \lambda_i \in \{1, -1\} \quad 1 \le i \le D$$

The result follows. $\square$

Observations

For a vertex x of $H(D,2)$:

- For $0 \le i \le D$
$$|\Gamma_i(x)| = \binom{D}{i}$$

- For $1 \le i \le D$ each vertex in $\Gamma_i(x)$ is adjacent to exactly i vertices in $\Gamma_{i-1}(x)$.

- For $0 \le i \le D-1$ each vertex in $\Gamma_i(x)$ is adjacent to exactly $D-i$ vertices in $\Gamma_{i+1}(x)$

- For $0 \le i \le D$ each vertex in $\Gamma_i(x)$ is adjacent to 0 vertices in $\Gamma_i(x)$

- For $0 \le i \le D$,
$$A E_i^* V \subseteq E_{i-1}^* V + E_{i+1}^* V \qquad E_i^* = E_i^*(x)$$

DEF 16 With above notation,
define a diagonal matrix

$$A^* = A^*(x) \in Mat_X(\mathbb{F})$$

with (y,y)-entry

$$(A^*)_{yy} = D - 2\,\partial(x,y) \qquad y \in X$$

Observe:

- $$A^* = \sum_{i=0}^{D} \theta_i^* E_i^* \qquad \theta_i^* = D-2i$$

- The distinct eigenvalues of A^* are
$$\{\theta_i^*\}_{i=0}^{D}$$

- For $0 \le i \le D$,
$$E_i^* V = \text{eigenspace of } A^* \text{ for } \theta_i^*$$

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Next goal: investigate how A, A^* are related

LEM 17 For vertices y, z of $H(D,2)$ with

$\partial(x,y) = \partial(x,z)$ and $\partial(y,z) = 2$,

we have

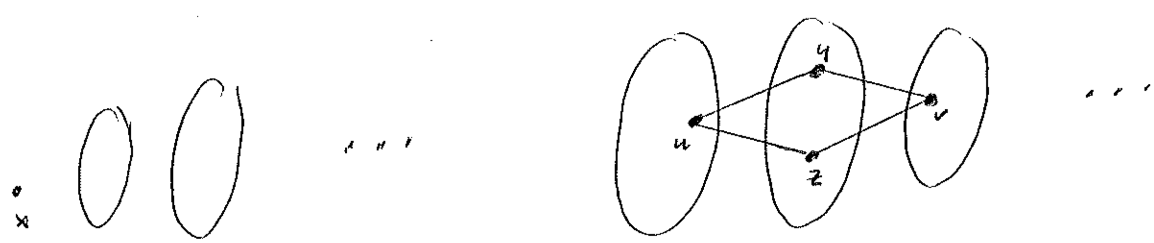
(i) $\exists$ unique vertex u of $H(D,2)$ such that

$\partial(y,u)=1$, $\partial(z,u)=1$, $\partial(x,u) = \partial(x,y)-1$

(ii) $\exists$ unique vertex v of $H(D,2)$ such that

$\partial(y,v)=1$, $\partial(z,v)=1$, $\partial(x,v) = \partial(x,y)+1$

pf Routine



Prop 18 Fix a vertex x of $H(D,2)$ and write $A^* = A^*(x)$

Then

(i) $A^2A^* - 2AA^*A + A^*A^2 = 4A^*$;

(ii) $(A^*)^2A - 2A^*AA^* + A(A^*)^2 = 4A$.

We will prove this in next lecture.