Math 846

Lecture 42

Our next goal is to explain how $\boxtimes_q$ is related to $U_q(\widehat{sl_2})$.

We will start with a more basic algebra $U_q(sl_2)$. $\mathbb{F}$ arb. $0 \neq q \in \mathbb{F}$ $q^2 \neq 1$

Def 34 Let $U_q(sl_2)$ denote the associative $\mathbb{F}$-algebra with 1 that has generators

$$e, f, k, k^{-1}$$

and relations

$$kk^{-1} = k^{-1}k = 1$$

$$ke = q^2 ek$$

$$kf = q^{-2} fk$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

<u>Note</u> the above definition is often called the Chevalley presentation.

To get a feel for $U_q(sl_2)$ it is helpful to work out the finite-dim'l irred modules.

I will describe these and leave the proof as an exercise.

Lem 35 Assume $\mathbb{F}$ is alg closed, and q is not a root of unity. Up to isomorphism the f.d. irred $U_q(sl_2)$-modules are

$$L(d, \varepsilon) \qquad d = 0, 1, 2, \ldots \qquad \varepsilon \in \{1, -1\}$$

$L(d, \varepsilon)$ has a basis $\{v_i\}_{i=0}^{d}$ st

$$kv_i = \varepsilon q^{d-2i} v_i \qquad (0 \le i \le d)$$
$$fv_i = [i+1]_q v_{i+1} \qquad (0 \le i \le d-1) \qquad fv_d = 0$$
$$ev_i = \varepsilon [d-i+1]_q v_{i-1} \qquad (1 \le i \le d), \qquad ev_0 = 0$$

(If $\mathrm{char}(\mathbb{F}) = 2$ view $\{1, -\}$ as having single element)

pf ex. $\square$

Def 36 Referring to Lem 35, we call ε the *type* of the module.

We now give another presentation of $U_q(sl_2)$, said to be *equitable*.

Lem 37 The algebra $U_q(sl_2)$ is isomorphic to the $\mathbb{F}$-algebra with gens

$$x, x^{-1}, y, z$$

and relations

$$x x^{-1} = x^{-1} x = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1$$

An isomorphism with the presentation in Def 34 is given by

$$x^{\pm 1} \to k^{\pm 1}$$

$$y \to k^{-1} + f$$

$$z \to k^{-1} - k^{-1} e f (q - q^{-1})^2.$$

The inverse isomorphism sends

$$k^{\pm 1} \to x^{\pm 1}$$

$$f \to y - x^{-1}$$

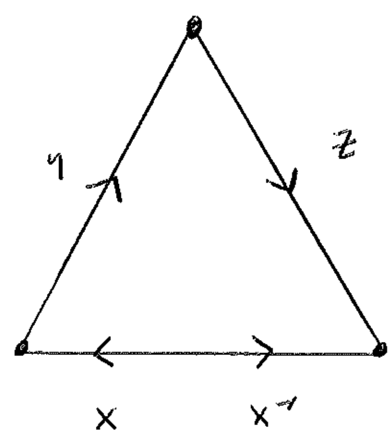
$$e \to (1 - xz) q^{-1} (q - q^{-1})^{-2}$$

pf One checks each map above is a hom of $\mathbb{F}$-algebras. One checks these maps are inverses. Therefore these maps are isomorphisms. $\square$

Def 38 We call x, x^{-1}, y, z the <u>equitable generators</u> of $U_q(sl_2)$.

A diagram for $U_q(sl_2)$:

Represent each equitable generator by a directed arc



Read off the relations using the conventions for $\boxtimes$, Lem 39 For $i \in \mathbb{Z}_4$ $\exists$ $\mathbb{F}$-algebra hom $U_q(\mathfrak{sl}_2) \to \boxtimes_q$ that sends

$$x \to x_{i,i+2}$$

$$x^{-1} \to x_{i+2,i}$$

$$y \to x_{i+2,i+3}$$

$$z \to x_{i+3,i}$$

pf Routine. $\square$

Next we define $U_q(\hat{\mathfrak{sl}}_2)$. Roughly speaking this algebra is generated by 2 copies of $U_q(\mathfrak{sl}_2)$ that are glued together in a certain way.

DEF 40 Let $U_q(\hat{sl}_2)$ denote the assoc $\mathbb{F}$-algebra with 1, with generators

$$e_i^+, \ e_i^-, \ k_i, \ k_i^{-1} \qquad (i=0,1)$$

and relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_0 k_1 = k_1 k_0$$

$$k_i e_i^{\pm} k_i^{-1} = q^{\pm 2} e_i$$

$$k_i e_j^{\pm} k_i^{-1} = q^{\mp 2} e_j \qquad i \neq j$$

$$[e_i^+, e_i^-] = \frac{k_i - k_i^{-1}}{q - q^{-1}} \qquad [r,s] = rs - sr$$

$$[e_0^{\pm}, e_1^{\mp}] = 0$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0 \qquad i \neq j$$

(q-Serre rels)

" Chevalley Presentation "

We now give the equitable presentation for $U_q(\widehat{sl_2})$

Lem 41 The $\mathbb{F}$-algebra $U_q(\widehat{sl_2})$ is isomorphic to the $\mathbb{F}$-algebra with gens

$$x_i, \; x_i^{-1}, \; y_i, \; z_i \qquad (i=0,1)$$

and the following relations:

$$x_i x_i^{-1} = x_i^{-1} x_i = 1$$

$$x_0 x_1 \quad \text{central}$$

$$\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} = 1, \qquad \frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} = 1,$$

$$\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} = 1, \qquad \frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} = x_0^{-1} x_1^{-1} \quad i \neq j$$

$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0 \quad i \neq j$$

$$z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0 \quad i \neq j$$

An iso with the presentation in Def 40

sends

$$x_i^{\pm 1} \longrightarrow k_i^{\pm 1}$$

$$y_i \longrightarrow k_i^{-1} + e_i^-$$

$$z_i \longrightarrow k_i^{-1} - k_i^{-1} e_i^+ q (q-q^{-1})^2$$

The inverse iso sends

$$k_i^{\pm 1} \longrightarrow x_i^{\pm 1}$$

$$e_i^- \longrightarrow y_i - x_i^{-1}$$

$$e_i^+ \longrightarrow (1 - x_i z_i) q^{-1} (q-q^{-1})^{-2}$$

pf Check each map is $\mathbb{F}$-alg hom, and the maps are inverses. Therefore each map is an $\mathbb{F}$-algebra iso. $\square$

Let $U_q(L(sl_2))$ denote the quotient of $U_q(\hat{sl}_2)$ by the 2-sided ideal generated by $X_0 X_1 - 1$ (in the equitable pres)

The algebra $U_q(L(sl_2))$ is called the

<u>$U_q(sl_2)$ loop algebra</u>

The following result shows how $U_q(L(sl_2))$ is related to $\boxtimes_q$.

LEM 42. The $\mathbb{A}$-algebra $U_q(L(\mathfrak{sl}_2))$

is iso to the algebra with gens

$$x_i,\ y_i,\ z_i \qquad (i=0,1)$$

and relations

$$x_0 x_1 = x_1 x_0 = 1$$

$$\frac{q x_i y_i - q^{-1} y_i x_i}{q-q^{-1}} = 1, \qquad \frac{q y_i z_i - q^{-1} z_i x_i}{q-q^{-1}} = 1,$$

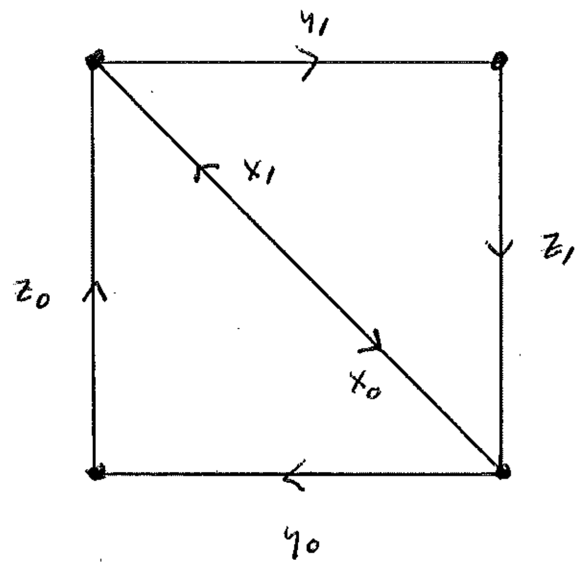
$$\frac{q z_i x_i - q^{-1} x_i z_i}{q-q^{-1}} = 1, \qquad \frac{q z_i y_j - q^{-1} y_j z_i}{q-q^{-1}} = 1 \qquad i \neq j$$

$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0 \qquad i \neq j$$

$$z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0 \qquad i \neq j$$

pf ex. $\square$

Diagram for $U_q(L(sl_2))$



Same conventions as for $\boxtimes_q$

thm 43 For $i \in \mathbb{Z}_4$ $\exists$ $\mathbb{F}$-alg hom

$U_q(L(sl_2)) \longrightarrow \boxtimes_q$ that reads

$$X_1 \longrightarrow X_{i,i+2} \qquad Y_1 \to X_{i+2,i+3} \qquad Z_1 \to X_{i+3,i}$$

$$X_0 \longrightarrow X_{i+2,i} \qquad Y_0 \to X_{i,i+1} \qquad Z_0 \to X_{i+1,i+2}$$

pf clear. $\Box$

Composing the canonical hom $U_q(\widehat{sl_2}) \to U_q(L(sl_2))$ with the hom in thm 43, we get an algebra hom $U_q(\widehat{sl_2}) \longrightarrow \boxtimes_q$. Let V denote an $\boxtimes_q$ module. Pulling back the $\boxtimes_q$-module structure to $U_q(\widehat{sl_2})$ via the above hom, V becomes a $U_q(\widehat{sl_2})$ module. In particular, the $\boxtimes_q$ module from thm 33 becomes a $U_q(\widehat{sl_2})$ module.

Cor 44 Referring to our TD system Φ on V

the vector space V becomes a $U_q(\hat{sl}_2)$-module on which the equitable gens act as follows.

gen	x_0	x_0^{-1}	y_0	z_0	x_1	x_1^{-1}	y_1	z_1
action	K^{-1}	K	A	B	K	K^{-1}	A^*	B^*

Also, the vector space V becomes a $U_q(\hat{sl}_2)$-module on which the equitable gens act as follows:

gen	x_0	x_0^{-1}	y_0	z_0	x_1	x_1^{-1}	y_1	z_1
action	$(K^*)^{-1}$	K^*	B	A^*	K^*	$(K^*)^{-1}$	B^*	A

Both $U_q(\hat{sl}_2)$-modules above are irreducible.

$\mathbb{F} = \mathbb{C}$

Consider a DRG $\Gamma = (X, R)$ with $D \geq 2$

and classical parameters (D, b, α, σ),

$$\alpha = b - 1$$

Fix $0 \neq q \in \mathbb{C}$ st

$$q^2 = b.$$

Fix $x \in X$ and write $T = T(x)$.

It turns out that A, A^* satisfy the

q-Serre relations (after an affine transformation).

Using Cor 44 we can turn the standard module

into a $U_q(\widehat{sl_2})$-module. This is explained in

the paper

T. Ito, P. Terwilliger. Distance-regular graphs and the q-tetrahedron algebra. European J. Combin. 30 (2009) 682–697.

THE END

$\square$