Math 846

Lecture 41

We continue to discuss the TD system

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

on V. Recall four flags on V from Def 20:

$$[0], \quad [D], \quad [0^*], \quad [D^*]$$

In the tetrahedron diagram we see six decompositions of V.

We now give these decomps more convenient names.

Given an ordered pair of distinct flags from Def 20, $[\alpha]$, $[\beta]$ let $[\alpha, \beta]$ denoted the associated decomp. Note that $[\beta, \alpha]$ is the inversion of $[\alpha, \beta]$ (ie reverse order)

We have

decomp	ith subspace of the decomp
$[0, D]$	$E_i V$
$[0^*, D^*]$	$E_i^* V$
$[0^*, D]$	$(E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V)$
$[0^*, 0]$	$(E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_{d-i} V)$
$[D^*, 0]$	$(E_d^* V + \cdots + E_{d-i}^* V) \cap (E_0 V + \cdots + E_{d-i} V)$
$[0^*, D]$	$(E_d^* V + \cdots + E_{d-i}^* V) \cap (E_i V + \cdots + E_d V)$

We now summarize the action of A and A^* on the six decomps.

LEM 22 Let $\{W_i\}_{i=0}^{d}$ denote a decomp of V from the above table. Then for $0 \le i \le d$ the action of A and A^* on W_i is:

Name	A action	A^* action
$[0, D]$	$(A - \theta_i I) W_i = 0$	$A^* W_i \le W_{i-1} + W_i + W_{i+1}$
$[0^*, D^*]$	$A W_i \le W_{i-1} + W_i + W_{i+1}$	$(A^* - \theta^*_i I) W_i = 0$
$[0^*, D]$	$(A - \theta_i I) W_i \le W_{i+1}$	$(A^* - \theta^*_i I) W_i \le W_{i-1}$
$[0^*, 0]$	$(A - \theta_{d-i} I) W_i \le W_{i+1}$	$(A^* - \theta^*_i I) W_i \le W_{i+1}$
$[D^*, 0]$	$(A - \theta_{d-i} I) W_i \le W_{i+1}$	$(A^* - \theta^*_{d-i} I) W_i \le W_{i-1}$
$[D^*, D]$	$(A - \theta_i I) W_i \le W_{i+1}$	$(A^* - \theta^*_{d-i} I) W_i \le W_{i-1}$

Pf Rows $[O, D]$ and $[O^*, D^*]$ restate def of TD system.

Row $[O^*, D]$ is from thm 9 (ii).

Remaining rows are from thm 9 (ii) applied to the relations of Φ. $\square$

Consider our six decomps of V from the tetrahedron picture.

The decomps $[0,D]$ and $[0^*,D^*]$ are the eigenspace decomps of A and A^*, resp.

Tempting to view remaining four decomps as eigenspace decomps.

To make progress here, we assume until further notice

$$0 \neq q \in \mathbb{F} \qquad q^2 \neq 1$$

$$\theta_i = q^{2i-d} \qquad 0 \le i \le d$$

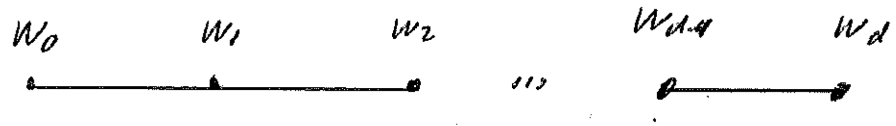
$$\theta_i^* = q^{d-2i} \qquad 0 \le i \le d$$

obs $q^{2i} \neq 1 \quad (1 \le i \le d)$ since $\theta_0, \theta_1, \ldots, \theta_d$ are mut distinct. In this case the TD rels become

$$A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 = 0$$

$$(A^*)^3 A - [3]_q (A^*)^2 A A^* + [3]_q A^* A (A^*)^2 - A (A^*)^3 = 0$$

<u>Notation</u> Given a decomp $\{W_i\}_{i=0}^{d}$ of V:

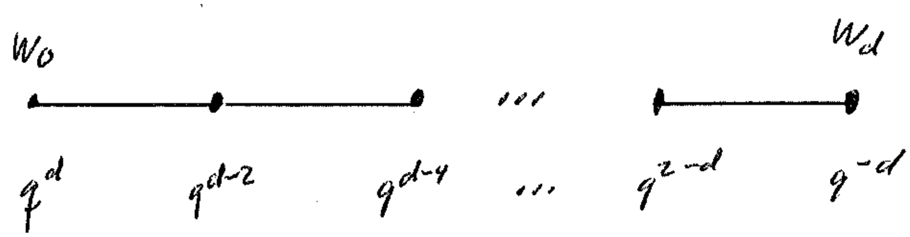


Consider the linear trans $T: V \to V$ st

for $0 \le i \le d$,

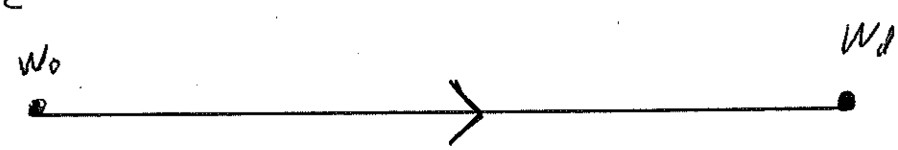
- W_i is an eigenspace of T with eigvalue q^{d-2i}

So



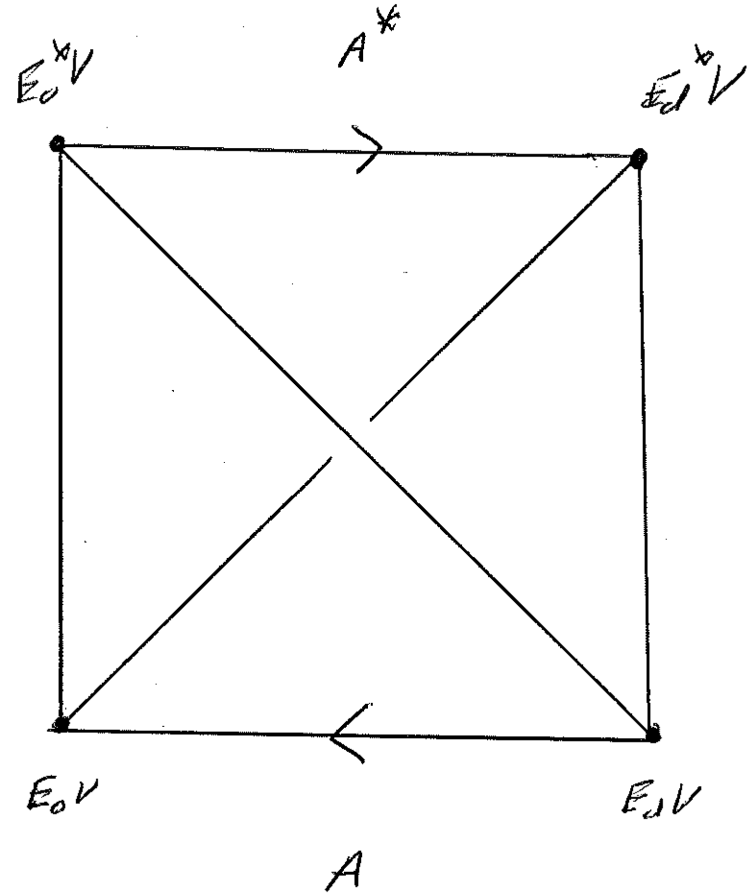
We often represent this lin trans by directed

arc



the inverse of T is represented by

Ex



Def 23 We define linear trans B, B^*, K, K^* on V as follows

$$
\begin{tikzcd}
E_0^*V \arrow[rr, "A^*"] & & E_d^*V \arrow[dd, "B^*"] \\
 & & \\
E_0V \arrow[uu, "B"] \arrow[uurr, "K^*"] & & E_dV \arrow[ll, "A"] \arrow[uull, "K"']
\end{tikzcd}
$$

So for example, for $0 \le i \le d$

$$(E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV)$$

is an eigenspace for K with eigenvalue q^{2i-d}

Next goal: find the relations satisfied by

$$A, A^*, B, B^*, K, K^*, K^{-1}, K^{*-1}.$$

We will use the following handy facts.

For any linear trans $\Psi : V \to V$ and $\forall\, \theta \in \mathbb{F}$

define $V_\Psi(\theta) = \{ v \in V \mid \Psi v = \theta v \}$

LEM 24 Given any linear trans

$\Psi : V \to V$ and $Z : V \to V$

Given $0 \neq \theta \in \mathbb{F}$ TFAE:

(i) $$\frac{q\Psi Z - q^{-1} Z \Psi}{q - q^{-1}} = I \quad \text{on } V_\Psi(\theta)$$

"q-Weyl equation"

(ii) $$(Z - \theta^{-1} I)\, V_\Psi(\theta) \subseteq V_\Psi(q^{-2}\theta)$$

pf $\forall\, v \in V_\Psi(\theta)$ we have

$$\left(q\Psi Z - q^{-1} Z \Psi - (q - q^{-1}) I\right) v$$
$$= q\left(\Psi - q^{-2}\theta I\right)\left(Z - \theta^{-1} I\right) v.$$

$\square$

LEM 25 Given lin trans. $Y : V \to V$
and $Z : V \to V$.
Given $0 \neq \theta \in \mathbb{F}$ TFAE:

(i) $$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I \text{ on } V_Z(\theta)$$

(ii) $$(Y - \theta^{-1} I) V_Z(\theta) \subseteq V_Z(q^2 \theta)$$

pf Replace (Y, Z, q) by (Z, Y, q^{-1})
in Lem 24. $\square$

LEM 26 Given lin trans $Y : V \to V$ and $Z : V \to V$ TFAE

(i) $$Y^3 Z - [3]_q Y^2 Z Y + [3]_q Y Z Y^2 - Z Y^3 = 0$$ on $V_Y(\theta)$

(ii) $$Z V_Y(\theta) \subseteq V_Y(q^2\theta) + V_Y(\theta) + V_Y(q^{-2}\theta)$$

pf ex. $\square$

LEM 27 We have

(i) $$\frac{qAB - q^{-1}BA}{q-q^{-1}} = I$$

(ii) $$\frac{qBA^{\vee} - q^{-1}A^{\vee}B}{q-q^{-1}} = I$$

(iii) $$\frac{qA^{\vee}B^{\vee} - q^{-1}B^{\vee}A^{\vee}}{q-q^{-1}} = I$$

(iv) $$\frac{qB^{\vee}A - q^{-1}AB^{\vee}}{q-q^{-1}} = I$$

pf In each case, combine L24/L25 and Lem 22. $\square$

LEM 28 We have

(i) $$\frac{qK^{-1}A - q^{-1}AK^{-1}}{q-q^{-1}} = I$$

(ii) $$\frac{qKA^{\vee} - q^{-1}A^{\vee}K}{q-q^{-1}} = I$$

(iii) $$\frac{qAK^{\vee} - q^{-1}K^{\vee}A}{q-q^{-1}} = I$$

(iv) $$\frac{qA^{\vee}K^{\vee-1} - q^{-1}K^{\vee-1}A^{\vee}}{q-q^{-1}} = I$$

pf In each case, combine L24/L25 and Lem 22. $\square$

We now give the actions of the B, B^* on the 6 decomps.

LEM 29 Let $\{W_i\}_{i=0}^{d}$ denote a decomp of V from the ⊠ picture. For $0 \le i \le d$ the action of B and B^* on W_i is:

Name	B-action	B^*-action
$[0, D]$	$(B - q^{d-2i}I)W_i \subseteq W_{i+1}$	$(B^* - q^{d-2i}I)W_i \subseteq W_{i+1}$
$[0^*, D^*]$	$(B - q^{2i-d}I)W_i \subseteq W_{i-1}$	$(B^* - q^{2i-d}I)W_i \subseteq W_{i-1}$
$[0^*, D]$	$(B - q^{2i-d}I)W_i \subseteq W_{i-1}$	$(B^* - q^{d-2i}I)W_i \subseteq W_{i+1}$
$[0^*, 0]$	$(B - q^{2i-d}I)W_i = 0$	$B^*W_i \subseteq W_{i-1} + W_i + W_{i+1}$
$[D^*, 0]$	$(B - q^{2i-d}I)W_i \subseteq W_{i+1}$	$(B^* - q^{d-2i}I)W_i \subseteq W_{i-1}$
$[D^*, D]$	$BW_i \subseteq W_{i-1} + W_i + W_{i+1}$	$(B^* - q^{d-2i}I)W_i = 0$

pf $[0,0]$, $[0^*,0^*]$, $[0^*,D]$, $[D^*,0]$

Routine using Lem 25, 26, 27, 28

$[0^*,0]$: $(B - q^{2i-d} I) W_i = 0$ by def of B.

Find $B^* W_i$:

View $W_i = (W_0 + \cdots + W_i) \cap (W_i + \cdots + W_d)$

$$B^* W_i \subseteq B^*(W_0 + \cdots + W_i)$$
$$= B^*(E_0^* V + \cdots + E_i^* V)$$
$$\subseteq E_0^* V + \cdots + E_{i+1}^* V \qquad \text{by row } [0^*,0^*]$$
$$= W_0 + \cdots + W_{i+1}$$

$$B^* W_i \subseteq B^*(W_i + \cdots + W_d)$$
$$= B^*(E_0 V + \cdots + E_{d-i} V)$$
$$\subseteq E_0 V + \cdots + E_{d-i+1} V \qquad \text{by row } [0,D]$$
$$= W_{i-1} + \cdots + W_d$$

So $B^* W_i \subseteq (W_0 + \cdots + W_{i+1}) \cap (W_{i-1} + \cdots + W_d)$

$= W_{i-1} + W_i + W_{i+1}$.

$[D^*,D]$ Sim.

$\square$

Lem 30 We have

(i) $$\frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} = I,$$

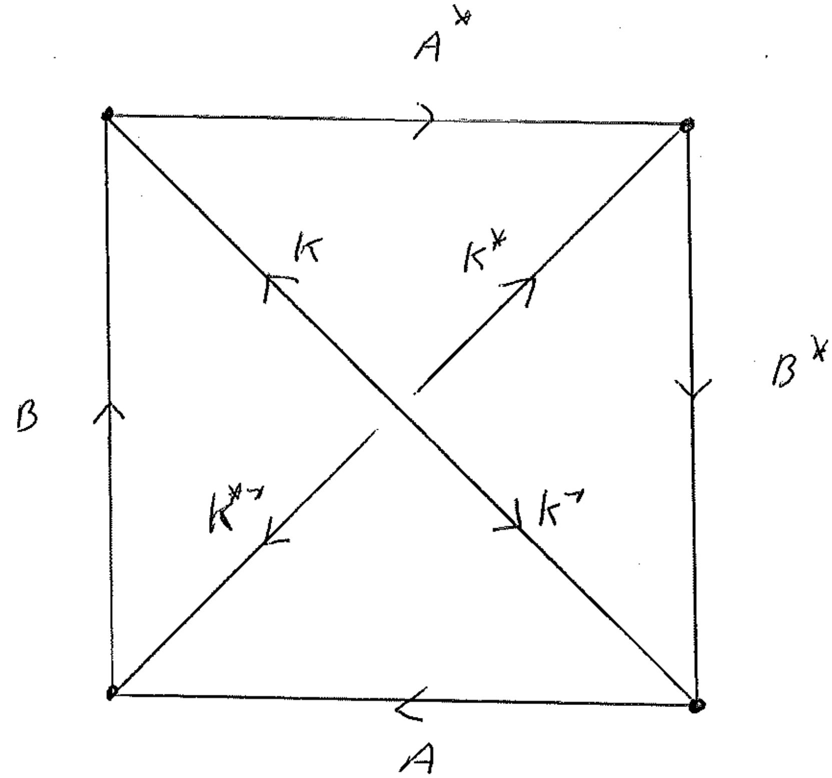
(ii) $$\frac{qB^{*}K - q^{-1}KB^{*}}{q - q^{-1}} = I,$$

(iii) $$\frac{qK^{*-1}B - q^{-1}BK^{*-1}}{q - q^{-1}} = I,$$

(iv) $$\frac{qK^{*}B^{\vee} - q^{-1}B^{\vee}K^{\vee}}{q - q^{-1}} = I$$

pf In each case, use Lem 25/26 and Lem 29. $\square$

the q-Weyl relations in summary



For each configuration

we have

$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I$$

A few more relations

LEM 31 We have

(i) $B^3B^* - [3]_q B^2B^*B + [3]_q BB^*B^2 - B^*B^3 = 0,$

(ii) $(B^*)^3B - [3]_q (B^*)^2 B B^* + [3]_q B^*B(B^*)^2 - B(B^*)^3 = 0.$

pf use Lem 26. $\square$

We now interpret our results so far in terms of an algebra

$\boxtimes_q$ "q-tetrahedron algebra"

Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ = cyclic group order 4

Def 32 let $\boxtimes_q$ denote the algebra with gens

$$\{ x_{ij} \mid i,j \in \mathbb{Z}_4 \quad i-j=1 \text{ or } i-j=2 \}$$

and relations

(i) For $i,j \in \mathbb{Z}_4$ st $i-j=2$

$$x_{ij} x_{ji} = 1$$

(ii) For $i,j,k \in \mathbb{Z}_4$ st the pair $(j-i, k-j)$ is one of

$(1,1)$ $(1,2)$, $(2,1)$

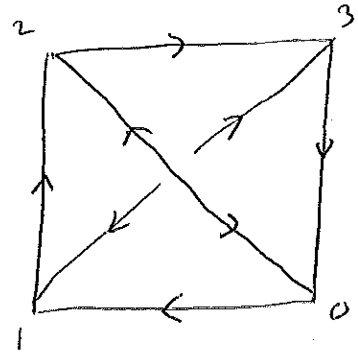
$$\frac{q x_{ij} x_{jk} - q^{-1} x_{jk} x_{ij}}{q-q^{-1}} = I$$

(iii) For $i,j,k,\ell \in \mathbb{Z}_4$ st $j-i=k-j=\ell-k=1$

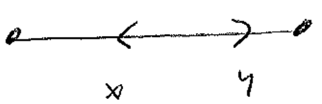
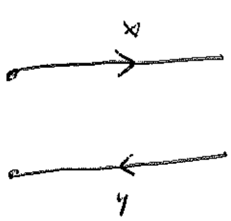
$$x_{ij}^3 x_{k\ell} - [3]_q x_{ij}^2 x_{k\ell} x_{ij} + [3]_q x_{ij} x_{k\ell} x_{ij}^2 - x_{k\ell} x_{ij}^3 = 0$$

Diagram for $\boxtimes_q$

Represent gen x_{ij} by 

Generators for $\boxtimes_q$: 

We read off the rels as follows

picture	meaning
	$xy = yx = 1$
	$\frac{qxy - q^{-1}yx}{q-q^{-1}} = 1$
	$x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = 0$

The following result is now immediate

Thm 33 Referring to our TD system Φ,
$\exists$ unique $\boxtimes_q$-module structure on V
st the generators act as follows

gen	X_{01}	X_{12}	X_{23}	X_{30}	X_{02}	X_{13}	X_{20}	X_{31}
action	A	B	A^*	B^*	K	K^*	K^{-1}	K^{*-1}

This $\boxtimes_q$-module is irreducible $\square$

— o —

Next goal: explain how $\boxtimes_q$
is related to $U_q(\hat{sl}_2)$