

Combinatorics 475

Lecture 40

We now give some examples using Polya thm.

Recall

$$X = \{1, 2, \dots, n\}$$

G = permutation group on X

Cycle index of G is polynomial

$$P_G(z_1, z_2, \dots, z_n) = \frac{\sum_{f \in G} \text{mon}(f)}{|G|}$$

For $f \in G$

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

$e_k = \#$ k -cycles in cycle factorization of f

Fix $p \geq 1$

dist colors $c_1, c_2, \dots, c_p$

corresp variables $u_1, u_2, \dots, u_p$

Polya thm

Generating function for colorings of X up to G -equivalence

$$= P_G(u_1 + u_2 + \dots + u_p, u_1^2 + u_2^2 + \dots + u_p^2, \dots, u_1^n + u_2^n + \dots + u_p^n)$$

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This means the following:

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For each monomial

$$u_1^{n_1} u_2^{n_2} \dots u_p^{n_p}$$

in X the following are the same:

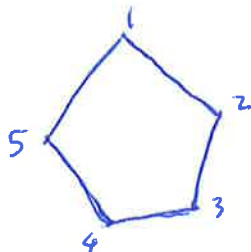
(i) the coefficient of $u_1^{n_1} u_2^{n_2} \dots u_p^{n_p}$

(ii) Up to G -equivalence, the number of colorings of X with

n_1	vertices colored	c_1
n_2	--	c_2

n_p	..	c_p

Ex Consider regular 5-gon



$$X = \{1, 2, 3, 4, 5\}$$

$G =$ gp of symmetries

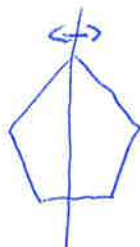
$=$ dihedral group of order 10

$$= \{p^i\}_{i=0}^4 \cup \{p^i \circ \tau\}_{i=0}^4$$

view as perm
gp on X

$p =$ clockwise 72° rotation

$\tau =$ reflection



Find P_G

f	cycle factorization of f	$mm(f)$
1		z_1^5
p		z_5
p^2		z_5
p^3		z_5
p^4		z_5
r	• ---	$z_1 z_2^2$
pr	' ---	$z_1 z_2^2$
$p^2 r$	' ---	$z_1 z_2^2$
$p^3 r$	' ---	$z_1 z_2^2$
$p^4 r$	' ---	$z_1 z_2^2$

$$P_G(z_1, z_2, z_3, z_4, z_5) =$$

$$z_1^5 + 5 z_1 z_2^2 + 4 z_5$$

Take $p=3$ colors

c_i	Red	White	Blue
u_i	r	w	b

Find coeff Polya generating function

Sol:

$$P_G(r+w+b, r^2+w^2+b^2, r^3+w^3+b^3, r^4+w^4+b^4, r^5+w^5+b^5)$$

=

$$\frac{(r+w+b)^5 + 5(r+w+b)(r^2+w^2+b^2)^2 + 4(r^5+w^5+b^5)}{10}$$

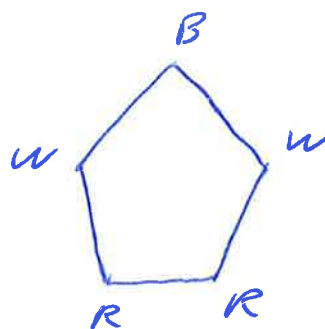
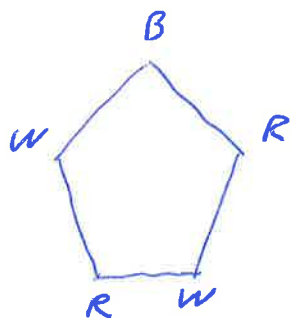
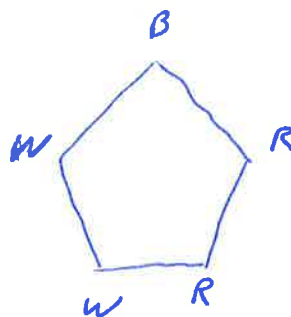
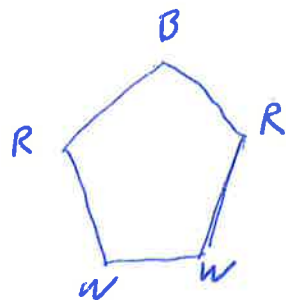
Up to G -equivalence, find the number of ways to color the vertices in X such that

2 vertices colored Red
 2 --- White
 1 --- Blue

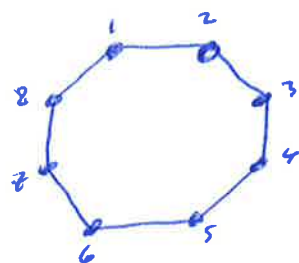
Sol: In Polya generating function

$$\begin{aligned} \text{coeff of } r^2 w^2 b &= \frac{\binom{5}{221} + 5 \times 2}{10} \\ &= 4 \end{aligned}$$

The colorings are



Consider regular 8-gon

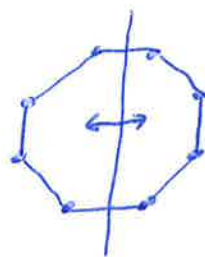


$$X = \{1, 2, \dots, 8\}$$

$G =$ group of symmetries
 $=$ dihedral group order 16
 $= \{p^i\}_{i=0}^7 \cup \{p^i \circ \tau\}_{i=0}^7$

$p = 45^\circ$ clockwise rot

$\tau =$ reflection



Find P_G

f	cycle factorization of f	$m(f)$
1		z_1^8
p		z_8
p^2	 	z_4^2
p^3		z_8 z_2^4
p^4	-----	
p^5		z_8
p^6	 	z_4^2
p^7		z_8
τ	-----	z_2^4
$p\tau$	..-----	$z_1^2 z_2^3$
$p^2\tau$	-----	z_2^4
$p^3\tau$	..-----	$z_1^2 z_2^3$
$p^4\tau$	-----	z_2^4
$p^5\tau$	..-----	$z_1^2 z_2^3$
$p^6\tau$	-----	z_2^4
$p^7\tau$	..-----	$z_1^2 z_2^3$

$$P_G(z_1, \dots, z_8) = \frac{z_1^8 + 4z_1^2 z_2^3 + 5z_2^4 + 2z_4^2 + 4z_8}{16}$$

Up to G -equiv, how many ways to color
the vertices in X with colors

Red, white, Blue?

Sol Set $z_i = 3$ for $1 \leq i \leq 8$

$$P_G(3, 3, \dots, 3) = \frac{3^8 + 4 \cdot 3^5 + 5 \cdot 3^4 + 2 \cdot 3^2 + 4 \cdot 3}{16}$$

$$= 166 \times 3$$

$$= 498$$



Ex

let $n = \text{prime number}$

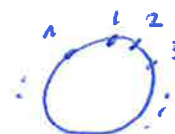
Fix $p \geq 1$, Dist Colors $c_1, \dots, c_p$

How many different necklaces can be made using

n beads with above colors?

let $X = \text{set of bead locations} = \{1, 2, \dots, n\}$

Here symmetry γ is



$G = \text{dihedral } \gamma \text{ of order } 2n$

$$= \{ \rho^i \}_{i=0}^{n-1} \cup \{ \rho^i \circ \gamma \}_{i=0}^{n-1}$$

$\rho = \text{clockwise rotation thru } \frac{360}{n} \text{ degrees}$

$\gamma = \text{reflection}$

Find P_G

f	cycle factorization	$\text{mm}(f)$
I	$\dots$	z_1^n
ρ^i $i \in \{1, \dots, n-1\}$	 $n\text{-cycle}$	z_n
$\rho^i \circ \gamma$ $i \in \{1, \dots, n-1\}$		$z_1, z_2^{\frac{n-1}{2}}$

$$P_G(z_1, \dots, z_n) = \frac{z_1^n + (n-1)z_n + n z_1 z_2^{\frac{n-1}{2}}}{2n}$$

necklaces

$$= \# \text{ ways to color } X \text{ with colors } c_1, c_2, \dots, c_p \\ (\text{up to } G\text{-equivalence})$$

$$= P_G(p, p, \dots, p)$$

$$= \frac{p^n + (n-1)p + n p^{\frac{n+1}{2}}}{2n}$$

Now for $p=2$ colors Red, Blue

Find the Polya gen function

c_i	Red	Blue
u_i	r	b

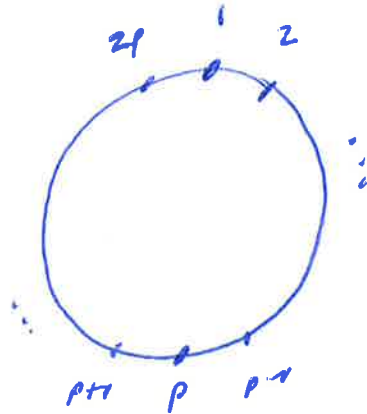
Polya Gen function is

$$P_G(r+b, r^2+b^2, \dots, r^n+b^n)$$

$$= \frac{(r+b)^n + (n-1)(r^n+b^n) + n(r+b)(r^2+b^2)^{\frac{n-1}{2}}}{2n}$$

Ex Find the cycle index for the
dihedral group D_{2p} p prime
(q has order $4p$) $\overset{G}{=}$

Sol View G as corner-symmetry group of regular $2p$ -gon

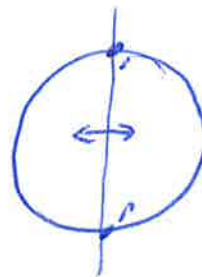


$$G = \{p^i\}_{i=0}^{n-1} \cup \{p^i\gamma\}_{i=0}^{n-1}$$

$$n = 2p$$

p : clockwise rot thru $\frac{360}{n}$

γ : reflection



f	cycle factorization of f	$\text{mon}(f)$
I		z_1^n
p^i i odd $1 \leq i \leq n-1$	 n -cycle	z_n
p^i i even $1 \leq i \leq n-1$	 two p -cycles	z_p^2
$p^i \gamma$ i even $0 \leq i \leq n-1$		$z_1^2 z_2^{p-1}$
$p^i \gamma$ i odd $0 \leq i \leq n-1$		z_2^p

$$P_G(z_1, \dots, z_n) = \frac{z_1^{2p} + p z_2^p + (p-1) z_1^2 + p z_1^2 z_2^{p-1} + p z_2^p}{4p}$$

Ex Find the Polya gen function for the
 number of different necklaces that can be
 made with $2p$ beads and two colors Red, Blue
 $p = \text{prime}$

Sol. In prev problem replace

$$z_i = r^i + b^i \quad 1 \leq i \leq 2p$$

Polya Gen Function is

$$\frac{(r+b)^{2p} + p(r^{2p} + b^{2p}) + (p-1)(r^p + b^p)^2 + p(r+b)^2(r^2+b^2)^{p-2} + p(r^2b^2)^p}{4p}$$

The End



