Math 846

Lecture 40

$\mathbb{F}$ arb field

V is nonO fd vector space over $\mathbb{F}$

$\Phi = (A, \{E_i\}_{i=0}^d ; A^*; \{E_i^*\}_{i=0}^d)$

is a TD system on V with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$

Def 10 The $\underline{\Phi\text{-split decomposition}}$ of V is the sequence $\{U_i\}_{i=0}^d$ from thm 9.

Next goal: for the Φ-split decomp $\{U_i\}_{i=0}^d$ show

- $E_i^* V$, U_i, $E_i V$ have same dim $(=\rho_i)$
- $\rho_i = \rho_{d-i}$ for $0 \le i \le d$
- $\rho_{i-1} \le \rho_i$ for $1 \le i \le d/2$

DEF 11 For $0 \le i \le d$ define $\mathbb{F}$-linear map $F_i : V \to V$ by

$$(F_i - I) U_i = 0$$

$$F_i U_j = 0 \text{ if } i \neq j \qquad (0 \le j \le d)$$

So F_i is the projection onto U_i

Define $F_{-1} = 0$ and $F_{d+1} = 0$

We have

$$F_i F_j = \delta_{ij} F_i \qquad (0 \le i, j \le d)$$

$$I = \sum_{i=0}^{d} F_i$$

$$U_i = F_i V \qquad (0 \le i \le d)$$

LEM 12 For $0 \leq i < j \leq d$,

(i) $E_i F_j = 0$,

(ii) $F_i E_j = 0$,

(iii) $E_j^* F_i = 0$,

(iv) $F_j E_i^* = 0$.

pf (i) Obs

$$
\begin{aligned}
E_i F_j V &= E_i U_j \\
&\subseteq E_i (U_j + \cdots + U_d) \\
&= E_i (E_j V + \cdots + E_d V) \\
&= 0
\end{aligned}
$$

(ii) Obs

$$
\begin{aligned}
F_i E_j V &\subseteq F_i (E_j V + \cdots + E_d V \\
&= F_i (U_j + \cdots + U_d) \\
&= 0
\end{aligned}
$$

(iii), (iv) Sim. $\square$

LEM 13 For $0 \le i \le d$,

(i) $F_i E_i F_i = F_i$

(ii) $E_i F_i E_i = E_i$

(iii) $F_i E_i^* F_i = F_i$

(iv) $E_i^* F_i E_i^* = E_i^*$

pf (i)

$$F_i = F_i^2$$
$$= F_i \left(E_0 + E_1 + \cdots + E_d \right) F_i$$

$$\left[\text{for } 0 \le j \le d, \quad \begin{array}{l} F_i E_j = 0 \quad \text{if } i < j \text{ and} \\ E_j F_i = 0 \quad \text{if } j < i \end{array} \right]$$

$$= F_i E_i F_i.$$

(ii)–(iv) Sim.

$\square$

LEM 14 For $0 \le i \le d$

(i) the $\mathbb{F}$-linear maps

$$\begin{aligned} U_i &\to E_i V \\ v &\to E_i v \end{aligned} \qquad\qquad \begin{aligned} E_i V &\to U_i \\ v &\to F_i v \end{aligned}$$

are bijections, and moreover they are inverses.

(ii) the $\mathbb{F}$-linear maps

$$\begin{aligned} U_i &\to E_i^* V \\ v &\to E_i^* v \end{aligned} \qquad\qquad \begin{aligned} E_i^* V &\to U_i \\ v &\to F_i v \end{aligned}$$

are bijections, and moreover they are inverses.

pf (i) They are inverses by Lem 13 (i), (ii).
It follows they are bijections.

(ii) Sim. $\square$

COR 15 For $0 \le i \le d$ the dimensions of E_iV, U_i, E_i^*V are equal. Denoting this dimension by ρ_i we have

$$\rho_i = \rho_{d-i}$$

pf By Lem 14 the dimensions of E_iV, U_i, E_i^*V are equal. Call this dimension ρ_i

To show $\rho_i = \rho_{d-i}$, suffices to show

$$\dim E_i^*V = \dim E_{d-i}^*V \qquad (*)$$

We just showed

$$\dim E_iV = \dim E_i^*V$$

Applying this result to $\Phi^{\Downarrow}$ we get

$$\dim E_iV = \dim E_{d-i}^*V.$$

This gives $(*)$. $\square$

Def 16 Set

$$R = A - \sum_{h=0}^{d} \theta_h F_h$$

$$L = A^* - \sum_{h=0}^{d} \theta_h^* F_h$$

We call R (resp L) the raising (resp lowering) map for Φ.

LEM 17 For $0 \le i \le d$ the following hold on U_i:

$$R = A - \theta_i I, \qquad L = A^* - \theta_i^* I$$

pf Since F_h is the projection onto U_h for $0 \le h \le d$. $\square$

COR 17 For $0 \le i \le d$,

(i) $R\, U_i \subseteq U_{i+1}$,

(ii) $L\, U_i \subseteq U_{i-1}$.

pf Combine thm 9 (ii) and Lem 17. $\square$

LEM 18 For $0 \le i \le j \le d$ the $\mathbb{F}$-linear map

$$U_i \longrightarrow U_j$$
$$v \rightarrow R^{j-i} v$$

is an injection if $i+j \le d$, a bijection if $i+j=d$, and a surjection if $i+j \ge d$.

The $\mathbb{F}$-linear map

$$U_j \longrightarrow U_i$$
$$v \longrightarrow L^{j-i} v$$

is an injection if $i+j \ge d$, a bijection if $i+j=d$, and a surjection if $i+j \le d$.

(Caution: above maps are not inverses, even if $i+j=d$)

pf Concerning R:

Case $i+j \le d$: Given $v \in U_i$ st $R^{j-i} v = 0$.

Show $v = 0$,

Obs

$$0 = R^{d-i} v$$

$$= (A - \theta_{d-1} I) \cdots (A - \theta_{i+1} I)(A - \theta_i I) v$$

So

$$v \in E_i V + \cdots + E_{d-1} V$$

$$\subseteq E_0 V + \cdots + E_{d-1} V.$$

So

$$v \in U_i$$

$$\subseteq U_0 + \cdots + U_i$$

$$= E_0^* V + \cdots + E_i^* V.$$

So

$$v \in \underbrace{(E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_{d-1} V)}_{\substack{\parallel \text{ by Lem 8 applied to } \Phi^{\Downarrow} \\ 0}}$$

So $v = 0$ ✓

Case $i + j = d$. U_i, U_j have same dim so above injection is bijection.

Case $i+j \geq d$: Given $w \in U_j$ find

$v \in U_i$ st $R^{j-i} v = w$.

Consider map

$$\begin{aligned} U_{d-j} &\longrightarrow U_j \\ u &\longrightarrow R^{d-2j} u \end{aligned}$$

By above comments this is bijection. So $\exists\ u \in U_{d-j}$

st $R^{d-2j} u = w$.

Def $v = R^{i+j-d} u$.

Then $v \in U_i$ and $R^{j-i} v = w$.

Our assertions about R are proved.

The proof for L is similar. $\square$

COR 19 We have

$$\rho_{i+1} \le \rho_i \qquad 1 \le i \le d/2$$

pf The map

$$U_{i+1} \longrightarrow U_i$$
$$v \longrightarrow Rv$$

is injective by LEM 18 so

$$\underset{\rho_{i+1}}{\underset{\shortparallel}{\dim U_{i+1}}} \le \underset{\rho_i}{\underset{\shortparallel}{\dim U_i}}$$

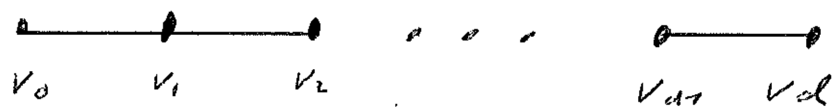
$\square$

Next goal: the tetrahedron diagram

Notation Given a decomp of V of length d:

$$\{V_i\}_{i=0}^{d}$$

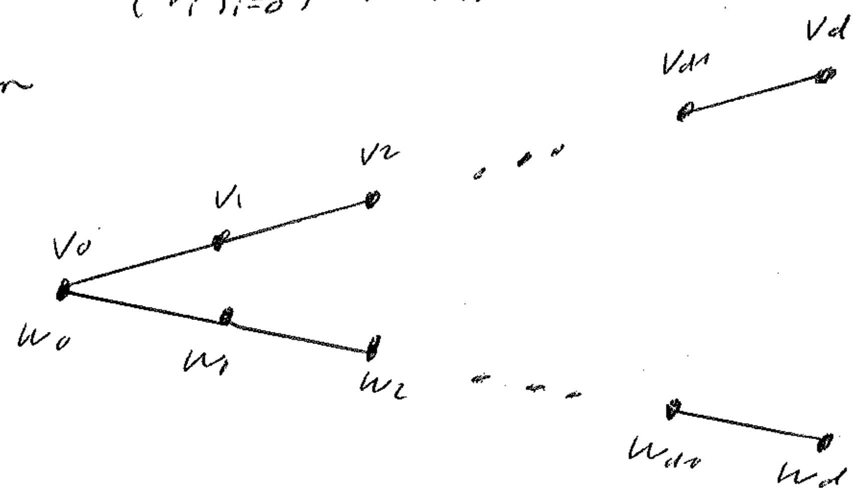
Represent by a dotted line segment:



Given two decomps of V of length d:

$$\{V_i\}_{i=0}^{d}, \quad \{W_i\}_{i=0}^{d}$$

then



means

$$V_0+V_1+\cdots+V_i = W_0+W_1+\cdots+W_i \qquad (0 \le i \le d)$$

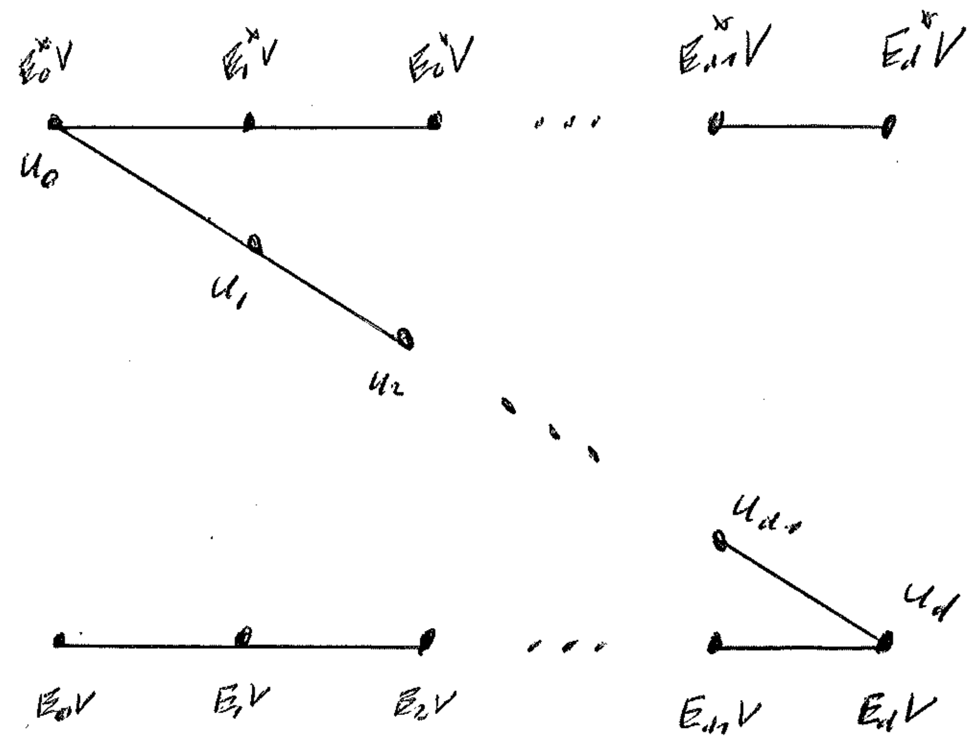
Recall the Φ-split decomp $\{U_i\}_{i=0}^d$ satisfies

$$U_0 + U_1 + \cdots + U_i = E_0^*V + E_1^*V + \cdots + E_i^*V$$

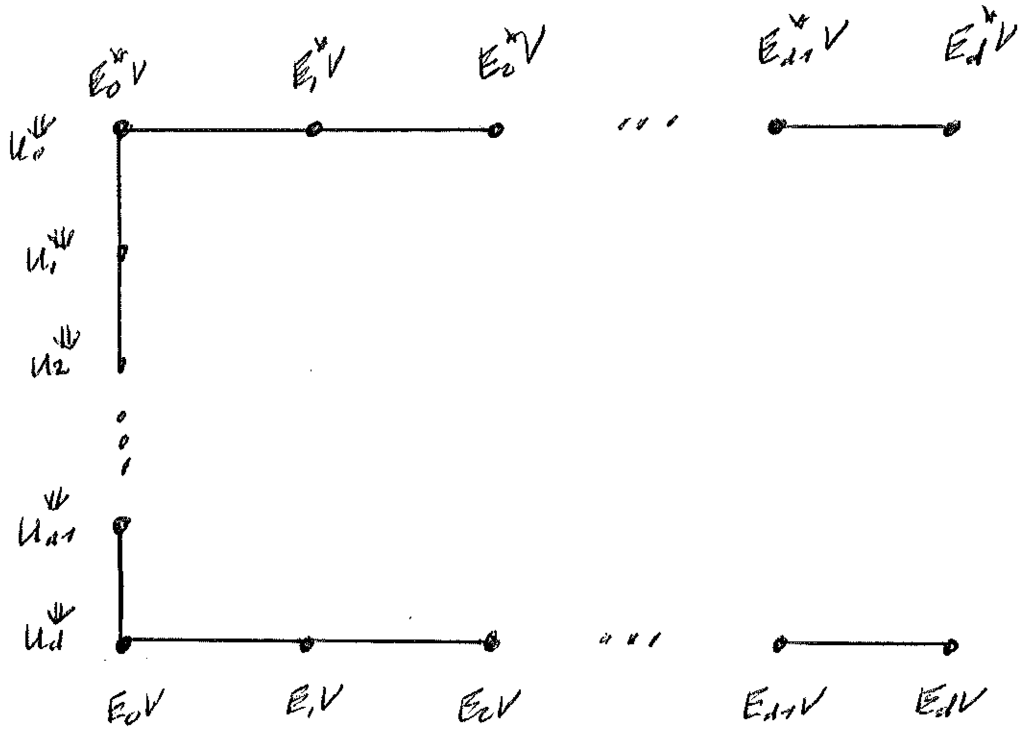
$$U_i + U_{i+1} + \cdots + U_d = E_iV + E_{i+1}V + \cdots + E_dV$$

for $0 \le i \le d$.

Corresponding diagram is:

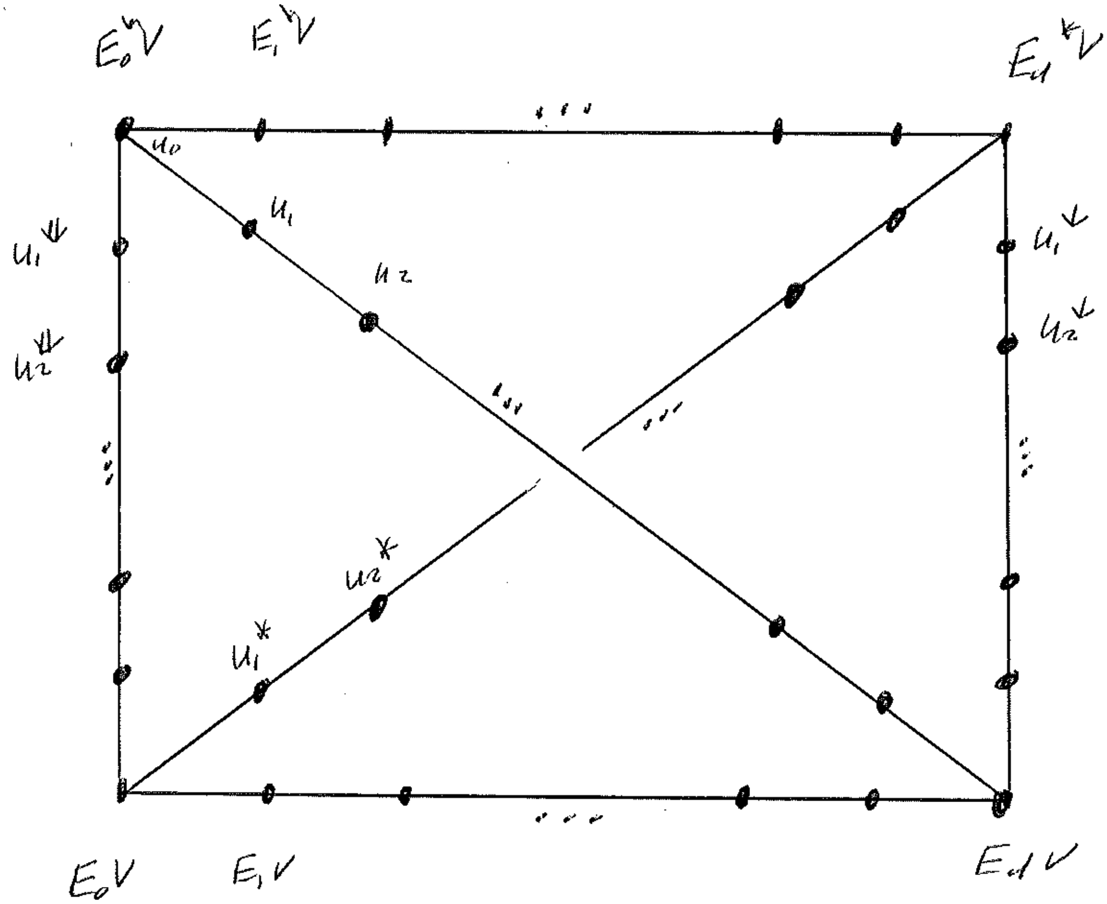


Applying this to $\Phi^{\Downarrow}$ get



Other relatives of Φ give similar diagrams

Altogether we get the following diagram.


$E_0^* V$
$E_1^* V$
$E_d^* V$
u_0
u_1
u_2
$u_1^{\Downarrow}$
$u_2^{\Downarrow}$
$u_1^{\downarrow}$
$u_2^{\downarrow}$
u_1^*
u_2^*
$E_0 V$
$E_1 V$
$E_d V$

We now describe the above diagram from a different pt of view.

<u>Notation</u> Let $\{a_i\}_{i=0}^d$ denote a sequence of positive integers whose sum is the dimension of V.

A <u>flag</u> on V of shape $\{a_i\}_{i=0}^d$ is a nested sequence of subspaces

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_d$$

st

$$\dim V_i = a_0 + a_1 + \cdots + a_i \qquad 0 \le i \le d.$$

So $V_d = V$.

Call V_i the ith component of the flag.

The following construction yields a flag on V of shape $\{s_i\}_{i=0}^d$:

Let $\{W_i\}_{i=0}^d$ denote a decomp of V with $s_i = \dim W_i$ for $0 \le i \le d$. Def

$$V_i = W_0 + W_1 + \cdots + W_i \qquad 0 \le i \le d.$$

Then $\{V_i\}_{i=0}^d$ is a flag on V of shape $\{s_i\}_{i=0}^d$.

Given 2 flags on V:

$$\{V_i\}_{i=0}^d \quad \text{and} \quad \{V_i'\}_{i=0}^d$$

Call these flags <u>opposite</u> whenever $\exists$ decomp $\{W_i\}_{i=0}^d$ of V s.t.

$$V_i = W_0 + \cdots + W_i \qquad 0 \le i \le d$$

$$V_i' = W_d + \cdots + W_{d-i} \qquad 0 \le i \le d.$$

In this case

$$V_i \cap V_j' = 0 \quad \text{if } i+j < d \quad (0 \le i,j \le d)$$

$$W_i = V_i \cap V_{d-i}' \qquad 0 \le i \le d$$

So $\{W_i\}_{i=0}^d$ is determined by the given flags. Call $\{W_i\}_{i=0}^d$ the <u>associated</u> decomp.

Def 20 Referring to our TD system Φ, we now define 4 flags on V, denoted

$$[0], \quad [D], \quad [0^*], \quad [D^*].$$

Each has shape $\{\rho_i\}_{i=0}^{d}$

flag	ith component
$[0]$	$E_0V + \cdots + E_iV$
$[D]$	$E_DV + \cdots + E_{D-i}V$
$[0^*]$	$E_0^*V + \cdots + E_i^*V$
$[D^*]$	$E_D^*V + \cdots + E_{D-i}^*V$

obs $[0], [D]$ are opp and $[0^*], [D^*]$ are opp.

LEM 21 The four flags in Def 20 are mutually opposite.

pf Show $[0^*]$, $[0]$ are opp

Take split decomp $\{U_i\}_{i=0}^d$ for Φ.

For $0 \le i \le d$

$$i\text{th comp of } [0^*] = E_0^* V + \cdots + E_i^* V$$
$$= U_0 + \cdots + U_i$$

$$i\text{th comp of } [0] = E_d V + \cdots + E_{d-i} V$$
$$= U_d + \cdots + U_{d-i}.$$

Rest of proof is similar. $\square$