

Combinatorics 475

Lecture 3

2.4 Permutations of multisets

We introduce the notion of a multiset with an example

Ex the letters in

MISSISSIPPI

form multiset

$$\{ I, I, I, I, M, P, P, S, S, S, S \}$$

which is abbreviated

$$\{ 4 \cdot I, 1 \cdot M, 2 \cdot P, 4 \cdot S \}$$

↑ ↑ ↑ ↑
"repetition numbers"

A multiset is like a set except its elements need not be distinct.

Ex letters in

ababab ...

form multiset

$$\{\infty \cdot a, \infty \cdot b\}$$

Def A multiset M is a set S
together with a function

$$R: S \rightarrow \{0, 1, 2, \dots\} \cup \infty$$

$\forall x \in S$ $R(x)$ is the repetition number of x

By the cardinality (a size) of M we mean

$$|M| = \sum_{x \in S} R(x)$$

EX How many ways can we order the letters in

MISSISSIPPI ?

This problem asks for the number of permutations
of the multiset $\{4 \cdot I, 1 \cdot M, 2 \cdot P, 4 \cdot S\}$

Size is $11 = 4 + 1 + 2 + 4$

To Create an ordering

Fill in 11 blanks

with
4 I's
1 M
2 P's
4 S's

We proceed in stages:

stage	to do	# choices
1	choose location for 4 I's	$\binom{11}{4}$
2	choose location for 1 M	$\binom{7}{1} = 7$ $11 - 4 = 7$
3	choose location for 2 P's	$\binom{6}{2}$ $7 - 1 = 6$
4	choose location for 4 S's	$\binom{4}{4} = 1$ $6 - 2 = 4$

By the multiplication principle

$$\begin{aligned}
 \# \text{ perms} &= \binom{11}{4} \binom{7}{1} \binom{6}{2} \cdot 1 \\
 &= \frac{11!}{4! 7!} \cdot \frac{7!}{1! 6!} \cdot \frac{6!}{2! 4!} \\
 &= \frac{11!}{4! 1! 2! 4!}
 \end{aligned}$$

size

repetition numbers

Then Given a multiset

$$M = \{ n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k \}$$

the number of permutations of M is

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

where

$$n = |M| = n_1 + n_2 + \dots + n_k$$

pf: just like above example

Def Given a multiset M

For an integer $r \geq 0$

An r -permutation of M is a sequence

$$a_1, a_2, \dots, a_r$$

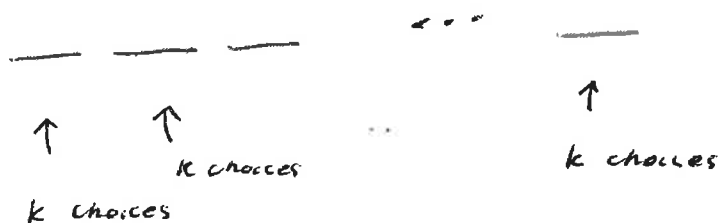
where $a_1, a_2, \dots, a_r$ are from M

Ex For the multiset

$$M = \{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_k\}$$

Find the number of r -perms for all $r \geq 0$

Sol. To create an r -perm fill in blanks



r blanks

of r -perms is k^r

Ex For the multiset

$$M = \{5 \cdot x, 7 \cdot y, 8 \cdot z\}$$

Find the number of 18-permutations of M .

Sol. size of M is 20

To obtain an 18-perm, remove 2 elements of M and order the remaining elements.

We partition the set of 18-perms according to what is removed

elements removed from M	# 18-perms
x, x	$\frac{18!}{3! \cdot 7! \cdot 8!}$
y, y	$\frac{18!}{5! \cdot 5! \cdot 8!}$
z, z	$\frac{18!}{5! \cdot 7! \cdot 6!}$
x, y	$\frac{18!}{4! \cdot 6! \cdot 8!}$
x, z	$\frac{18!}{4! \cdot 7! \cdot 7!}$
y, z	$\frac{18!}{5! \cdot 6! \cdot 7!}$

the # of 18-perms of M is sum of $\uparrow$

Ex Given finite set S $n = |S|$

Given labelled boxes

Box 1, Box 2, ..., Box k .

Distribute all the elements of S into boxes as follows:

Box 1 gets exactly n_1 elements

Box 2 gets exactly n_2 elements

...

Box k gets exactly n_k elements

$$(so \quad n_1 + n_2 + \dots + n_k = n)$$

How many ways to distribute?

Sol. Fill the boxes in stages

stage	to do	# choices
1	choose elements for Box 1	$\binom{n}{n_1}$
2	... 2	$\binom{n-n_1}{n_2}$
3	... 3	$\binom{n-n_1-n_2}{n_3}$
...	...	...
k	... k	$\binom{n-n_1-n_2-\dots-n_{k-1}}{n_k}$

$$\binom{n_k}{n_k} = 1$$

distributions =

$$\binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \dots$$

$$= \frac{n!}{n_1! (n-n_1)!} \frac{(n-n_1)!}{n_2! (n-n_1-n_2)!} \frac{(n-n_1-n_2)!}{n_3! (n-n_1-n_2-n_3)!} \dots$$

$$= \frac{n!}{n_1! n_2! \dots n_k!}$$

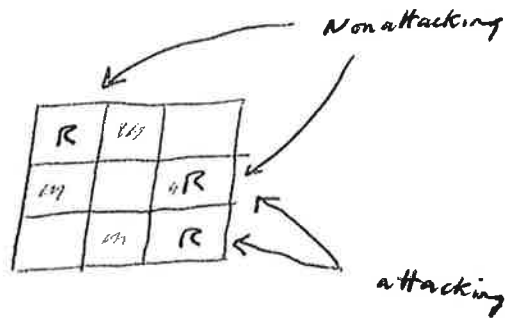
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Consider $n \times n$ chessboard

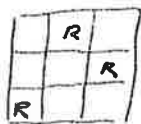
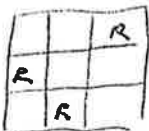
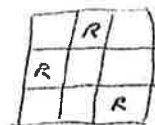
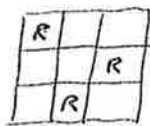
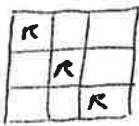
Recall in game of chess

rooks move horiz or vert

2 rooks are nonattacking if they are in distinct rows and distinct cols



Ex. In How many ways can we place 3 mutually nonattacking rooks on an 3×3 chessboard?



6 ways

Ex For $n \geq 1$ In how many ways
can we place n mutually nonattacking rooks
on an $n \times n$ chessbd?

Sol. Obs

For each rook placement on chessbd

each row gets unique rook

each col gets unique rook

For $1 \leq i \leq n$

let R_i = rook in row i

let c_i = number of column that contains R_i

Sequence

$c_1, c_2, \dots, c_n$

is permutation of $\{1, 2, \dots, n\}$

$C \times n = 3$

	R	
R		
		R

$c_1 = 2$
 $c_2 = 1$
 $c_3 = 3$

this gives bijection

$$\left\{ \begin{array}{l} \text{set of} \\ \text{rook placements} \\ \text{on chessbd} \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{l} \text{set of} \\ \text{permutations of} \\ \{1, 2, \dots, n\} \end{array} \right\}$$

$$\begin{aligned} \# \text{ rook placements on chessbd} &= \# \text{ perms of } \{1, 2, \dots, n\} \\ &= n! \end{aligned}$$

□

Ex Referring to previous example

Suppose we now color the rooks with k distinct colors
Require:

n_1 rooks get color 1
 n_2 ... 2
 $\vdots$ $\vdots$ $\vdots$
 n_k ... k

$$(n = n_1 + n_2 + \dots + n_k)$$

How many colored-rook arrangements are possible?

Sol We create a colored-rook arrangement in stages

stage	to do	# choices
1	put n mutually nonattacking uncolored rooks on chessboard	$n!$
2	color the rooks	$\frac{n!}{n_1! n_2! \dots n_k!}$

$$\# \text{ colored-rook arrangements} = \frac{(n!)^2}{n_1! n_2! \dots n_k!}$$

□