Math 846

Lecture 3

We continue to discuss a connected graph $\Gamma = (X, \mathcal{R})$. Fix $x \in X$ and write $M^* = M^*(x)$, $T = T(x)$, $d = d(x)$.

observe

- $\overline{B}^t \in M \quad \forall B \in M$
- $\overline{B}^t \in M^* \quad \forall B \in M^*$
- $\overline{B}^t \in T \quad \forall B \in T$

By a <u>T-module</u> we mean a subspace

$W \subseteq V$ such that

$$BW \subseteq W \qquad \forall B \in T$$

For example

$W = 0$ and $W = V$ are T-modules.

If W and U are T-modules, then so are

$$W \cap U, \qquad W + U$$

where $W + U = \mathrm{span}(w + u \mid w \in W, u \in U)$

A T-module W is irreducible whenever $W \neq 0$ and W contains no T-module besides $0, W$.

Our next goal: show that every T-module is an orthogonal direct sum of irreducible T-modules.

For a subspace $W \subseteq V$ define

$$W^{\perp} = \{ v \in V \mid \langle w, v \rangle = 0 \ \forall w \in W \}$$

"orthogonal complement of W in V"

By linear algebra

$$V = W + W^{\perp} \quad \text{(orthog direct sum)}$$

Next assume that W is a T-module.

Then $W^{\perp}$ is also a T-module.

check: $\forall B \in T$ and $\forall u \in W^{\perp}$

show $Bu \in W^{\perp}$

Given

$$\langle u, W \rangle = 0$$

show

$$\langle Bu, W \rangle \overset{?}{=} 0$$

obs

$$\langle Bu, W \rangle = \langle u, \underbrace{\overbrace{\overline{B}^t}^{\in T} W}_{\text{in } W} \rangle = 0 \quad \text{ok}$$

Prop 8 With above notation every T-module W is an orthogonal direct sum of irreducible T-modules.

pf Assume $W \neq 0$, else done.

WLOG assume the result is true for every T-module with dimension $< \dim W$.

W contains an irreducible T-module U (since $\dim W < \infty$)

observe

$$W = U + U^{\perp} \cap W \quad \text{(orthog direct sum)} \quad (*)$$

$U^{\perp}$ and W are T-modules, so $U^{\perp} \cap W$ is T-module.

Also
$$\dim W = \underset{\neq 0}{\dim U} + \dim(U^{\perp} \cap W)$$

So $\dim(U^{\perp} \cap W) < \dim W$

Now $U^{\perp} \cap W$ is an orthog direct sum of irred T-modules. Result follows in view of $(*)$. $\square$

For T-modules W, U by an <u>isomorphism of T-modules</u> from W to U we mean an $\mathbb{F}$-linear bijection $\sigma : W \to U$ such that on W,

$$\sigma B = B \sigma \qquad \forall B \in T$$

The T-modules W, U are called <u>isomorphic</u> whenever $\exists$ a T-module isomorphism from W to U.

— ∘ —

For a non 0 T-module W

For $0 \le i \le d$ obs

$$E_i^* W = W \cap E_i^* V$$

Moreover

W is an orthogonal direct sum of the non 0 subspaces among $\{E_i^* W\}_{i=0}^{d}$

By the <u>endpoint</u> of W we mean

$$\min \{ i \mid 0 \le i \le d, \ E_i^* W \ne 0 \}$$

By the <u>diameter of W</u> we mean

$$\left| \{ i \mid 0 \le i \le d, \ E_i^* W \ne 0 \} \right| - 1$$

Recall the primitive idempotents $\{E_i\}_{i=0}^r$ of Γ.

For $0 \le i \le r$

$$E_i W = W \cap E_i V$$

Moreover

W is the orthogonal direct sum of the non 0 subspaces among $\{E_i W\}_{i=0}^r$

By the <u>dual endpoint of W</u> (with respect to the given ordering $\{E_i\}_{i=0}^r$) we mean

$$\min \{ i \mid 0 \le i \le r, \ E_i W \ne 0 \}$$

By the <u>dual diameter of W</u> we mean

$$\left| \{ i \mid 0 \le i \le r, \ E_i W \ne 0 \} \right| - 1$$

The T-module W is called <u>thin</u> whenever

$$\dim E_i^* W \le 1 \qquad (0 \le i \le d)$$

The T-module W is called <u>dual thin</u> whenever

$$\dim E_i W \le 1 \qquad (0 \le i \le r)$$

Natural questions / projects:

- How does the standard module V decompose into an orthogonal direct sum of irred T-modules?
- Find the isomorphism classes of irred T-modules
- Describe each irred T-module W. In particular find a nice basis for W and give the action of M, M^* on that basis.
- Describe the center $Z(T)$
- Study the graphs Γ for which the irred T-modules are well behaved. For instance, when all the irred T-modules are thin and dual thin.
- How is the algebra T related to other algebras in the literature, such as Lie algebras and quantum groups?

These questions/projects are the main topic of the course.

Exercises I about the complete graph K_n $n \geq 3$

Recall spectrum $\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$

Fix $x \in X$ and write $T = T(x)$ etc.

Prove:

(i) $E_0 = n^{-1} J$ $\quad J =$ all 1's matrix in $\mathrm{Mat}_X(\mathbb{F})$

(ii) $n E_0 E_0^* E_0 = E_0$ and $n E_0^* E_0 E_0^* = E_0^*$

(iii) T has a basis

$I,\ E_0,\ E_0^*,\ E_0 E_0^*,\ E_0^* E_0$

(iv). the algebra T is isomorphic to

$\mathrm{Mat}_{2\times 2}(\mathbb{F}) \oplus \mathbb{F}$

(v) Up to isomorphism there exist exactly two irred T-modules. Both are thin. The first has dim 2, endpt 0, dual endpt 0, diameter 1, dual diam 1

the second has dim 1, endpt 1, dual endpt 1, diam 0, dual diameter 0.

(vi) the center $Z(T)$ has basis e_0, e_1 where

$$e_0 = \frac{n}{n-1}\left(E_0 + E_0^* - E_0 E_0^* - E_0^* E_0 \right), \quad e_1 = I - e_0$$

Moreover $e_0^2 = e_0$, $\quad e_1^2 = e_1$, $\quad e_0 e_1 = e_1 e_0 = 0$

We return to our connected graph $\Gamma = (X, R)$. Lec 10

We saw that T is generated by M and M^*

We now consider how M and M^* are related.

LEM 9 For $0 \le i \le d$,

$$A E_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V$$

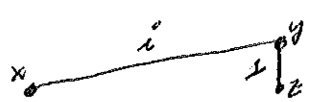
where $E_{-1}^* = 0$, $E_{d+1}^* = 0$.

pf Recall

$$E_i^* V = \mathrm{Span}(\hat{y} \mid y \in \Gamma_i(x))$$

For $y \in \Gamma_i(x)$,

$$A\hat{y} = \sum_{z \in \Gamma(y)} \hat{z}$$



$\partial(x, z) \in \{i-1, i, i+1\}$

$$\subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V$$

$\square$

Cor 10 With above notation,

$$E_i^* A E_j^* = 0 \quad \text{if } |i-j|>1 \qquad (0 \le i,j \le d)$$

pf Use LEM 9. $\square$

Compare M and M^*:

	M	M^*
basis	$\{E_i\}_{i=0}^r$	$\{E_i^*\}_{i=0}^d$
generator	A	?

Also COR 10

To complete the picture, we seek a matrix A^* that generates M^* and

$$E_i A^* E_j = 0 \quad \text{if } |i-j|>1 \qquad (0 \le i,j \le r)$$