

Combinatorics 475

Lecture 39

## 14.3 Cont.

Recall

Fix  $X = \{1, 2, \dots, n\}$   $n \geq 1$

$p \geq 1$

Fix colors  $c_1, c_2, \dots, c_p$

$C =$  set of all colorings of  $X$  with these colors

Fix

$G =$  perm gp on  $X$

Then  $\#$   $G$  equiv classes on  $C$

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

Burnside

$$= P_G(p, p, \dots, p)$$

where

$$P_G(z_1, z_2, \dots, z_n) = \frac{\sum_{f \in G} \text{mon}(f)}{|G|}$$

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

$e_k =$  #  $k$ -cycles in cycle factorization of  $f$   
( $1 \leq k \leq n$ )

Function  $P_G$  called cycle index of  $G$ .

Given  $c \in C$

For  $1 \leq i \leq p$  define

$$n_i = \# \text{ vertices in } X \text{ colored } c_i \text{ by } c$$

So

$$\left. \begin{aligned} n_1 + n_2 + \dots + n_p &= n = |X| \\ n_i &\geq 0 \quad 1 \leq i \leq p \end{aligned} \right\} *$$

Given a sol  $\{n_i\}_{i=1}^p$  of  $*$ , define

$C_{n_1, n_2, \dots, n_p}$  = set of colorings of  $X$  that have exactly  $n_i$  vertices colored  $c_i$  for  $1 \leq i \leq p$

Note that

$C_{n_1, n_2, \dots, n_p}$  is  $G$ -invariant

Problem Find

$$N(G, C_{n_1, n_2, \dots, n_p})$$

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Sol We give the generating function for  $**$

Introduce variables

$$u_1, u_2, \dots, u_p$$

( $u_i$  corresponds to color  $c_i$  for  $1 \leq i \leq p$ )

Gen function for  $**$  is

$$\sum_{n_1, n_2, \dots, n_p} N(G, c_{n_1, n_2, \dots, n_p}) u_1^{n_1} u_2^{n_2} \dots u_p^{n_p}$$



where sum is over all sets  $n_1, n_2, \dots, n_p$  to  $**$

thm (Polya counting formula)

With above notation

$$\star = P_G(u_1 + u_2 + \dots + u_p, u_1^2 + u_2^2 + \dots + u_p^2, \dots, u_1^n + u_2^n + \dots + u_p^n)$$

where  $P_G =$  cycle index of  $G$

pf (for  $p=2$ )

2 colors  $c_1, c_2$

show

$$\sum_{\substack{n_1 \geq 0 \\ n_2 \geq 0 \\ n_1 + n_2 = n}} N(G, C_{n_1, n_2}) u_1^{n_1} u_2^{n_2} = P_G(u_1 + u_2, u_1^2 + u_2^2, \dots, u_1^n + u_2^n)$$

Consider

$$P_G(u_1 + u_2, u_1^2 + u_2^2, \dots, u_1^n + u_2^n)$$

as a poly in  $u_1, u_2$ .

$$\forall n \quad n_1 \geq 0 \quad n_2 \geq 0$$

$$\underbrace{\text{coeff of } u_1^{n_1} u_2^{n_2}}_{\gamma}$$

Assume  $n_1 + n_2 = n$ , else  $\gamma = 0$  by constr.

Show

$$\gamma = N(G, C_{n_1, n_2})$$

↙ set of elements in  $C_{n_1, n_2}$  fixed by  $f$

By Burnside

$$N(G, C_{n_1, n_2}) = \frac{\sum_{f \in G} |C_{n_1, n_2}(f)|}{|G|}$$

Recall

$$P_G(z_1, z_2, \dots, z_n) = \frac{\sum_{f \in G} \text{mon}(f)}{|G|}$$

$\forall f \in G$  show

$$\text{contrib of } \text{mon}(f) \text{ to } \gamma = |C_{n_1, n_2}(f)|$$

$$m_m(f) = z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

$e_k = \#$   $k$ -cycles in cycle factorization of  $f$

Replace

$$z_1 \rightarrow u_1 + u_2$$

$$z_2 \rightarrow u_1^2 + u_2^2$$

$\vdots$

$$z_n \rightarrow u_1^n + u_2^n$$

$m_{\mathbb{Q}N}(f)$  becomes

$$(u_1 + u_2)^{e_1} (u_1^2 + u_2^2)^{e_2} \dots (u_1^n + u_2^n)^{e_n}$$

In this polynomial show

$$\text{coef of } u_1^{n_1} u_2^{n_2} = |C_{n_1, n_2}(f)|$$

expand

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$$(u_1 + u_2)^{e_1} (u_1^2 + u_2^2)^{e_2} \cdots (u_1^n + u_2^n)^{e_n}$$

=

$$\left( \sum_{l_1=0}^{e_1} \binom{e_1}{l_1} u_1^{l_1} u_2^{e_1-l_1} \right)$$

x

$$\left( \sum_{l_2=0}^{e_2} \binom{e_2}{l_2} u_1^{2l_2} u_2^{2e_2-2l_2} \right)$$

x

...

x

$$\left( \sum_{l_n=0}^{e_n} \binom{e_n}{l_n} u_1^{nl_n} u_2^{ne_n-nl_n} \right)$$

In the above polynomial the coeff of  $u_1^{n_1} u_2^{n_2}$  is

$$\sum \binom{e_1}{l_1} \binom{e_2}{l_2} \cdots \binom{e_n}{l_n}$$

$$l_1, l_2, \dots, l_n$$

$$0 \leq l_1 \leq e_1$$

$$0 \leq l_2 \leq e_2$$

⋮

$$0 \leq l_n \leq e_n$$

$$l_1 + 2l_2 + \cdots + nl_n = n_1$$

show this equals  $|C_{n_1, n_2}(f)|$

Find  $|C_{n_1, n_2}(f)|$

cycle factorization of  $f$ :

$$e_1 \left\{ \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right. \quad 1\text{-cycles}$$

$$e_2 \left\{ \begin{array}{c} \text{---} \\ \text{---} \\ \cdot \\ \text{---} \end{array} \right. \quad 2\text{-cycles}$$

$$e_3 \left\{ \begin{array}{c} \Delta \\ \Delta \\ \cdot \\ \Delta \end{array} \right. \quad 3\text{-cycles}$$

$$\vdots \quad \vdots \quad \vdots$$

To find  $|C_{n_1, n_2}(f)|$  we construct a coloring

$c \in C_{n_1, n_2}(f)$  in two steps

Step I decide how many

1 - cycles get colored  $c_1$   $(l_1)$   
 2 - cycles  $\dots$   $c_2$   $(l_2)$   
 $\vdots$   
 $\dots$   
 $n$  - cycles  $\dots$   $c_n$   $(l_n)$

One choice for each sol  $\{l_i\}_{i=1}^n$  to

$$0 \leq l_1 \leq c_1$$

$$0 \leq l_2 \leq c_2$$

$$\vdots$$

$$0 \leq l_n \leq c_n$$

$$l_1 + 2l_2 + \dots + nl_n = n,$$

For each choice  $\{l_i\}_{i=1}^n$  we proceed to step II :

Step II

| stage    | to do                                | # choices                                  |
|----------|--------------------------------------|--------------------------------------------|
| 1        | decide which 1-cycles to color $c_1$ | $\begin{pmatrix} c_1 \\ l_1 \end{pmatrix}$ |
| 2        | $\dots$ 2-cycles $\dots$             | $\begin{pmatrix} c_2 \\ l_2 \end{pmatrix}$ |
| $\vdots$ |                                      |                                            |
| n        | $\dots$ n-cycles $\dots$             | $\begin{pmatrix} c_n \\ l_n \end{pmatrix}$ |

So

$$|C_{n_1, n_2}(f)| = \sum_{\substack{0 \leq l_1 \leq e_1 \\ 0 \leq l_2 \leq e_2 \\ \vdots \\ 0 \leq l_n \leq e_n \\ l_1 + 2l_2 + \dots + nl_n = n_2}} \binom{e_1}{l_1} \binom{e_2}{l_2} \dots \binom{e_n}{l_n}$$

this proves Polya thm for  $p=2$

□

