Math 846

Lecture 39

Given a DRG $\Gamma = (X, R)$ diam D

Assume Γ is Q-poly wrt $\{E_i\}_{i=0}^{D}$.

Let W denote an irred T-module

with endpt r, dual endpt t, diam d, dual diam d^*

We will show:

- $d = d^*$
- for $0 \le i \le d$,

$$\dim E^*_{r+i} W = \dim E_{t+i} W$$

(call it ρ_i)

- the sequence $\{\rho_i\}_{i=0}^{d}$ is symmetric and unimodal; ie

$$\rho_i = \rho_{d-i} \qquad 0 \le i \le d,$$

$$\rho_{i-1} \le \rho_i \qquad 1 \le i \le d/2.$$

The above results hold for any tridiagonal pair. So we now turn to tridiagonal pairs.

CHAPTER 5 Tridiagonal pairs and $U_q(\hat{sl}_2)$

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Until further notice $\mathbb{F}$ is any field

We recall the definition of a tridiagonal pair.

Let V denote a non 0, finite-dim'l vector space over $\mathbb{F}$. A tridiagonal pair (TD pair) on V is an ordered pair of $\mathbb{F}$-linear maps

$$A: V \to V, \qquad A^*: V \to V$$

such that

(i) each of A, A^* is diagonalizable on V

(ii) $\exists$ an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A st

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d)$$

where $V_{-1} = 0$, $V_{d+1} = 0$

(iii) $\exists$ an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* st

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \le i \le \delta)$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V st
$AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$

Note Given a TD pair A, A^* on V. Then A^*, A is a TD pair on V. Call these TD pairs duals.

Note A Leonard pair is the same thing as a TD pair for which the V_i, V_i^* all have dimension 1.

Next we give an example of a TD pair related to quantum groups.

Pick $0 \neq q \in \mathbb{F}$ that is not a root of unity (so $q^n \neq 1$ for $n \geq 1$).

The algebra U_q^+ is defined by generators x, y and relations

$$x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = 0,$$
$$y^3x - [3]_q y^2xy + [3]_q yxy^2 - xy^3 = 0$$

"cubic q-Serre relations"

U_q^+ is called the <u>positive part of $U_q(\hat{sl_2})$</u>

Recall the above q-Serre relations are a special case of the tridiagonal relations.

For the next example assume $\mathbb{F}$ is alg closed.

Ex 1 Let V denote a finite-diml irred U_q^+-module on which neither x, y is nilpotent (so no positive power of x, y is 0).

Then x, y act on V as a TD pair.

pf $\forall \theta \in \mathbb{F}$ define

$$V(\theta) = \theta\text{-eigenspace of } x \text{ on } V$$
$$= \{ v \in V \mid xv = \theta v \}$$

Possibly $V(\theta) = 0$

We show that $\forall\, 0 \neq \theta \in \mathbb{F}$,

$$y V(\theta) \subseteq V(q^{-2}\theta) + V(\theta) + V(q^{2}\theta). \qquad (*)$$

To see $(*)$, pick $v \in V(\theta)$, so $xv = \theta v$. Obs

$$0 = \left(x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 \right) v$$

$$= x^3 y v - [3]_q \theta x^2 y v + [3]_q \theta^2 x y v - \theta^3 y v$$

$$= (x - q^{-2}\theta)(x-\theta)(x - q^2\theta) y v.$$

We assume q is not a root of 1, so

$q^{-2}\theta,\ \theta,\ q^2\theta$ mutually distinct

so

$$yv \in V(q^{-2}\theta) + V(\theta) + V(q^2\theta).$$

Since $\mathbb{F}$ is alg closed and x is not nilp on V, x has at least one non0 eigval θ on V.

Consider sequence

$$\theta,\ q^{-2}\theta,\ q^{-4}\theta, \ldots$$

these scalars are mut distinct since q is not a root of 1.

So they are not all eigenvalues of X on V.

So $\exists$ eigval α of X on V st $q^{-2}\alpha$ is not an eigval of X on V.

Consider sequence

$$\alpha, \quad q^2\alpha, \quad q^4\alpha, \ldots$$

$\exists$ $d \geq 0$ st

$$\alpha q^{2i} \text{ is eigval of } X \text{ on } V$$

for $0 \leq i \leq d$ but not $i = d+1$.

Set

$$V_i = V(\alpha q^{2i}) \qquad 0 \leq i \leq d$$

and $V_{-1} = 0$, $V_{d+1} = 0$

Obs

$$V_0 + V_1 + \cdots + V_d$$

is nm 0 and X-invariant by Constr.

By (*),

$$y V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d)$$

So $V_0 + V_1 + \cdots + V_d$ is y-invariant.

Now $V = \sum_{i=0}^{d} V_i$

by the irreducibity of V. Now x is diagonalizable on V.

Interchanging x, y in the above arguments, we find that x, y act on V as a TD pair $\square$

Farb. Given a TD pair A, A^* on V

An ordering of the eigenspaces $\{V_i\}_{i=0}^{d}$ of A is called <u>standard</u> whenever

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d)$$

If the ordering $\{V_i\}_{i=0}^{d}$ is standard then so is $\{V_{d-i}\}_{i=0}^{d}$ and no other ordering is standard.

An ordering of the primitive idempotents or eigenvalues of A is called standard whenever the corresp ordering of the eigenspaces is standard.

Similar comments apply to A^*.

Def 2 Let V = non 0 f.d. vectorspace / $\mathbb{F}$.

A <u>tridiagonal system</u> (TD system) on V is a sequence

$$\Phi = \left(A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta}\right)$$

st

(i) A, A^* is a TD pair on V

(ii) $\{E_i\}_{i=0}^{d}$ is a standard ordering of the prim idempotents of A

(iii) $\{E_i^*\}_{i=0}^{\delta}$ " " " A^*

We say Φ is <u>over $\mathbb{F}$</u>. Call V the <u>underlying vectorspace</u>

Ref to Def 2, obs

$$E_i A^* E_j = 0 \quad \text{if } |i-j|>1 \qquad (0 \le i,j \le d),$$
$$E_i^* A E_j^* = 0 \quad \text{if } |i-j|>1 \qquad (0 \le i,j \le \delta).$$

Next goal: explain the <u>relatives</u> of a TD system.

Given a TD system Φ on V as in Def 2.

Then the following are TD systems on V:

$$\Phi^* = \left(A^*, \{E_i^*\}_{i=0}^{\delta}, A, \{E_i\}_{i=0}^{d}\right) \qquad \text{"dual"}$$

$$\Phi^{\downarrow} = \left(A, \{E_i\}_{i=0}^{d}, A, \{E_{\delta-i}^*\}_{i=0}^{\delta}\right) \qquad \text{"1st inversion"}$$

$$\Phi^{\Downarrow} = \left(A, \{E_{d-i}\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta}\right) \qquad \text{"2nd inv"}$$

Viewing $*$, $\downarrow$, $\Downarrow$ as permutations on the set of all TD systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1,$$

$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow \Downarrow = \Downarrow \downarrow$$

The group generated by symbols $*$, $\downarrow$, $\Downarrow$ subject to the relations above, is the dihedral group D_4

This group is the group of symmetries of a square, and has 8 elements.

So $*$, $\downarrow$, $\Downarrow$ induce an action of D_4 on the set of all TD systems.

Two TD systems are called <u>relatives</u> whenever they are in the same orbit of this D_4-action.

DEF 3 Given TD system

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta})$$

For $0 \le i \le d$ let $\theta_i =$ eigenvalue of A for E_i

For $0 \le i \le \delta$ let $\theta_i^* =$ eigenvalue of A^* for E_i^*

call $\{\theta_i\}_{i=0}^{d}$ the <u>eigenvalue sequence</u> of Φ

call $\{\theta_i^*\}_{i=0}^{\delta}$ the <u>dual eigenvalue sequence</u> of Φ

Notation 4 Given TD system Φ.

Given $g \in D_4$.

For any object f associated with Φ, f^g will denote the corresp object for $\Phi^{g^{-1}}$. So

$$\theta_i(\Phi^*) = \theta_i^*(\Phi) \quad \text{etc.}$$

Next goal: split decomposition

__Notation__ let V = nonO, fd v.s/$\mathbb{F}$

d = nonneg integer

A __decomposition of V of length d__ is a

sequence of subspaces $\{U_i\}_{i=0}^{d}$ st

$U_i \neq 0$ $(0 \leq i \leq d)$ and

$$V = \sum_{i=0}^{d} V_i \quad \text{(dir sum)}$$

We set $U_{-1} = 0$, $U_{d+1} = 0$.

Given TD system

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta})$$

on V. We now show $d = \delta$. Also we show

$\exists$ unique decomp $\{U_i\}_{i=0}^{d}$ of V s.t

$$(A - \theta_i I) U_i \subseteq U_{i+1} \qquad 0 \leq i \leq d,$$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1} \qquad 0 \leq i \leq d$$

where $\{\theta_i\}_{i=0}^d$ (resp $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue sequence (resp dual eigenvalue sequence) of Φ

Def 5 For all integers i, j define

$$V_{ij} = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_jV + E_{j+1}V + \cdots + E_dV)$$

Interpret sum on left to be 0 if $i<0$ and V if $i>d$
right ----- $i>d$ $i<0$

...

LEM 6 We have

(i) $V_{i0} = E_0^*V + \cdots + E_i^*V \qquad (0 \le i \le d)$

(ii) $V_{dj} = E_jV + \cdots + E_dV \qquad (0 \le j \le d)$

$\square$

pf clear

Lem 7 For $0 \le i \le d$ and $0 \le j \le d$,

(i) $(A - \theta_j I) V_{ij} \subseteq V_{i+1, j+1}$

(ii) $A V_{ij} \subseteq V_{ij} + V_{i+1, j+1}$

(iii) $(A^* - \theta^*_i I) V_{ij} \subseteq V_{i-1, j-1}$

(iv) $A^* V_{ij} \subseteq V_{ij} + V_{i-1, j-1}$

pf (i) We have

$$(A - \theta_j I)(E^*_0 V + \cdots + E^*_i V) \subseteq E^*_0 V + \cdots + E^*_{i+1} V$$

and

$$(A - \theta_j I)(E_j V + \cdots + E_d V) = E_{j+1} V + \cdots + E_d V$$

(ii) By (i)

(iii), (iv) Similar

$\square$

LEM 8. We have $d = \delta$. Moreover

$$V_{ij} = 0 \text{ if } i < \delta \qquad (0 \le i, j \le d) \qquad (*)$$

pf First assume $\delta \le d$. We show $(*)$.

To do this, we show

$$V_{0,r} + V_{1,r+1} + \cdots + V_{d-r,d} \qquad (**)$$

is 0 for $0 < r \le d$.

Let r be given, let W be the sum $(**)$. By Lem 7 (iii), (iv)

$$AW \subseteq W, \quad A^*W \subseteq W$$

So $W = 0$ or $W = V$ by the irred of V.

By Def 5 each term in $**$ is contained in

$$E_r V + \cdots + E_d V$$

So $$W \subseteq E_r V + \cdots + E_d V$$

We assume $r > 0$ so $W \neq V$.

So $W = 0$.

We have shown $(**)$ is 0 for $0<r\le d$,
so $(*)$ holds

Show $d=\delta$. Suppose $d\neq\delta$, so $\delta<d$.

Set $i=\delta$, $j=d$ in $*$ to get

$$V_{\delta d}=0$$

By Lem 6

$$V_{\delta d}=E_d V$$

Contr. So $d=\delta$.

The case of $\delta\ge d$ is similar. $\square$

Thm 9 For any subspaces $\{U_i\}_{i=0}^d$ of V, TFAE:

(i) $U_i = (E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV)$ $0 \le i \le d$

(ii) $\{U_i\}_{i=0}^d$ is a decomp of V, and

$(A - \theta_i I)U_i \subseteq U_{i+1}$, $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$

for $0 \le i \le d$

(iii) For $0 \le i \le d$, both

$$U_i + \cdots + U_d = E_iV + \cdots + E_dV, \quad (1)$$

$$U_0 + \cdots + U_i = E_0^*V + \cdots + E_i^*V. \quad (2)$$

pf (i) → (ii) To get the inclusions set $i=j$ in Lem 8 and note that $U_i = V_{ii}$

Show $\{U_i\}_{i=0}^d$ is a decomp of V.

Show $U_0 + U_1 + \cdots + U_d = V$

Define $W = U_0 + U_1 + \cdots + U_d$. By the inclusions

$AW \subseteq W$ and $A^*W \subseteq W$

So $W = 0$ or $W = V$

Also W contains U_0 and

$U_0 = E_0^* V \neq 0$

So $W \neq 0$. So $W = V$.

Show the sum $V = U_0 + U_1 + \cdots + U_d$ is direct.

Suf to show $U_i \cap (U_0 + \cdots + U_{i-1}) = 0 \qquad 1 \leq i \leq d$

Let i be given. For $0 \leq j \leq i-1$

$U_j \subseteq E_0^* V + \cdots + E_j^* V \subseteq E_0^* V + \cdots + E_{i-1}^* V$

So $U_0 + U_1 + \cdots + U_{i-1} \subseteq E_0^* V + \cdots + E_{i-1}^* V$.

Also $U_i \subseteq E_i V + \cdots + E_d V$

So

$$U_i \cap (U_0 + \cdots + U_{i-1}) \subseteq (E_i V + \cdots + E_d V) \cap (E_0^* V + \cdots + E_{i-1}^* V)$$

$$= V_{i-1, i}$$

$$= 0$$

by Lem 8.

Show $U_i \neq 0$ for $0 \leq i \leq d$:

We have $U_0 = E_0^* V \neq 0$

$U_d = E_d V \neq 0$

Suppose $\exists i \; (1 \leq i \leq d-1)$ st $U_i = 0$.

Then $U_0 + \cdots + U_{i-1}$ is a nonzero, proper subspace of V that is invar under A, A^* Contr.

We have shown $\{U_i\}_{i=0}^d$ is a decomp of V.

(ii) → (iii) Show (1). Abbr

$W = U_i + \cdots + U_d$

$Z = E_i V + \cdots + E_d V$

Show $Z \subseteq W$: def

$$X = \prod_{h=0}^{i-1} (A - \alpha_h I)$$

obs

$$Z = XV$$

Also using the inclusions in (ii),

$$XV \subseteq W$$

So $Z \subseteq W$ ✓

Show $W \subseteq Z$: def

$$Y = \prod_{h=i}^{d} (A - \alpha_h I)$$

obs

$$Z = \{v \in V \mid Yv = 0\}$$

By the inclusions in (ii),

$$Y u_j = 0 \quad \text{for } i \le j \le d$$

So $YW = 0$

So $W \subseteq Z$.

We have shown $W = Z$ so (1) holds. (2) is similar.

(iii) → (i) First show the sum $U_0 + \cdots + U_d$ is direct. To do this, show

$$(U_0 + U_1 + \cdots + U_{i-1}) \cap U_i = 0 \qquad 1 \le i \le d$$

Let i be given. Obs

$$\begin{aligned}(U_0 + U_1 + \cdots + U_{i-1}) \cap U_i &\subseteq (U_0 + \cdots + U_{i-1}) \cap (U_i + \cdots + U_d) \\ &= (E_0^* V + \cdots + E_{i-1}^* V) \cap (E_i V + \cdots + E_d V) \\ &= V_{i-1,i} \\ &= 0.\end{aligned}$$

So $U_0 + \cdots + U_d$ is direct. Now

$$\begin{aligned}U_i &= (U_0 + \cdots + U_i) \cap (U_i + \cdots + U_d) \\ &= (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V)\end{aligned}$$

$\square$