

Combinatorics 475

Lecture 38

14.3 Polya Counting

the cycle factorization of a permutation

Given $X =$ nonempty finite set

say $X = \{1, 2, \dots, n\}$

Given integer r ($1 \leq r \leq n$)

Given mutually distinct $a_1, a_2, \dots, a_r \in X$

let

$$[a_1, a_2, \dots, a_r]$$

"cycle notation"

denote the perm of X that sends

$$a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \dots \rightarrow a_r$$


and fixes all other elements of X

Call this perm a cycle of order r

or

r -cycle

ex $n=8$

let $f = [1326]$

Write f in 2-line notation

Sol

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 2 & 4 & 5 & 1 & 7 & 8 \end{pmatrix}$$

Note Each cycle has multiple cycle notations.

For instance

$$[1326] = [3261] = [2613] = [6132]$$

Note cycle of order 1 is just the identity:

$$[i] = I \quad \text{for } 1 \leq i \leq n$$

Given two cycles:

$$f = [a_1, a_2, \dots, a_r]$$

$$g = [b_1, b_2, \dots, b_s]$$

Call these cycles disjoint whenever

$$a_i \neq b_j$$

$$1 \leq i \leq r,$$

$$1 \leq j \leq s$$

In this case f, g commute:

$$f \circ g = g \circ f$$

pf: obs

f fixes each of $b_1, b_2, \dots, b_s$

g fixes each of $a_1, a_2, \dots, a_r$

For $x \in X$ show

$$(f \circ g)(x) = (g \circ f)(x)$$

Case $x \in \{a_1, a_2, \dots, a_r\}$

say $x = a_i$

$$f \circ g: \quad a_i \xrightarrow{g} a_i \xrightarrow{f} a_{i+1}$$

)) -

$$g \circ f: \quad a_i \xrightarrow{f} a_{i+1} \xrightarrow{g} a_{i+1}$$

Case $x \in \{b_1, b_2, \dots, b_n\}$:

say $x = b_1$

$$f \circ g: b_1 \xrightarrow{g} b_{2n} \xrightarrow{f} b_{2n} \quad \text{"✓"}$$

$$g \circ f: b_1 \xrightarrow{f} b_1 \xrightarrow{g} b_{2n}$$

Case $x \notin \{a_1, \dots, a_r\} \quad x \notin \{b_1, \dots, b_n\}$:

$$f \circ g: x \xrightarrow{g} x \xrightarrow{f} x \quad \text{"✓"}$$

$$g \circ f: x \xrightarrow{f} x \xrightarrow{g} x$$

Ex Given any perm f of X

say $n = 8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 6 & 8 & 7 & 1 & 2 \end{pmatrix}$$

Express f as a product of disjoint cycles

↑

with respect to composition

f:

$$1 \rightarrow 4 \rightarrow 6 \rightarrow 7$$



$$2 \rightarrow 5 \rightarrow 8$$



3 ↻

$$f = [1467] \circ [258] \circ [3]$$

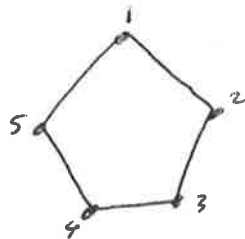
"cycle factorization of f "

In general

LEM Each permutation of X is a product of disjoint cycles

□

Ex Consider regular 5-gon



$$X = \{1, 2, 3, 4, 5\}$$

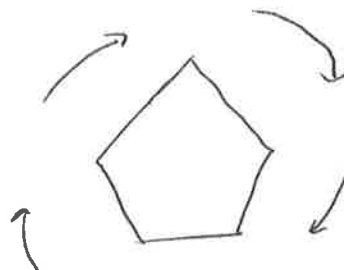
Let $G =$ group of symmetries
 $=$ dihedral gp of order 10

View as
 perm gp on X

$$G = \underbrace{\{p^i\}_{i=0}^4}_{\text{rot}} \cup \underbrace{\{p^i \tau\}_{i=0}^4}_{\text{refl}}$$

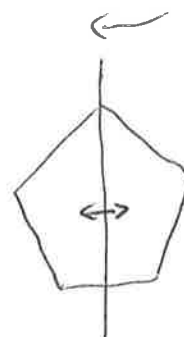
where

p is clockwise 72° rot
 p^5



τ is reflection

τ :



For each $f \in G$ find its cycle factorization

f	cycle factorization
I	$[1] \circ [2] \circ [3] \circ [4] \circ [5]$
ρ	$[1\ 2\ 3\ 4\ 5]$
ρ^2	$[1\ 3\ 5\ 2\ 4]$
ρ^3	$[1\ 4\ 2\ 5\ 3]$
ρ^4	$[1\ 5\ 4\ 3\ 2]$

f	desc	cycle factorization of f
τ		$[1][25][34]$
$\rho\tau$		$[12][35][4]$
$\rho^2\tau$		$[13][2][45]$
$\rho^3\tau$		$[14][23][5]$
$\rho^4\tau$		$[15][24][3]$

Given $X = \{1, 2, \dots, n\}$

Given perm f of X

Consider cycle factorization of f

Define $\#(f)$ = the number of cycles in this factorization

Ex For above 5-gon

f	$\#(f)$
I	5
ρ	1
ρ^2	1
ρ^3	1
ρ^4	1
τ	3
$\rho\tau$	3
$\rho^2\tau$	3
$\rho^3\tau$	3
$\rho^4\tau$	3

Given $X = \{1, 2, \dots, n\}$

Given integer $p \geq 1$

Given p distinct colors $c_1, c_2, \dots, c_p$

Let $C =$ set of all colorings of X with these colors

Given a permutation f of X

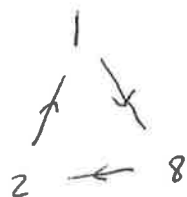
Find $|C(f)|$ in terms of $\#(f)$

$\uparrow$
pts in X fixed by f

Sol ex $n=8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 5 & 7 & 3 & 6 & 4 & 2 \end{pmatrix}$$

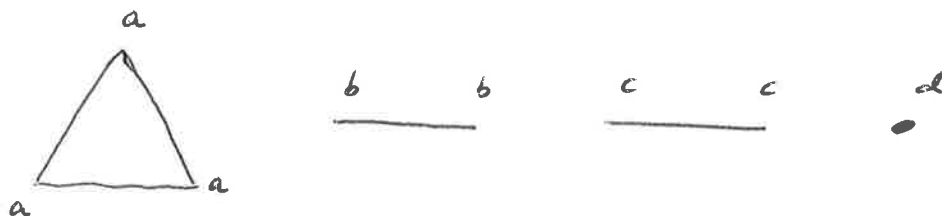
Find cycle factorization of f



$$f = [182] \circ [35] \circ [47] \circ [6]$$

$$\#(f) = 4$$

Describe $C(f)$



$$a, b, c, d \in \{c_1, c_2, \dots, c_r\}$$

$$\# \text{ choices for } a, b, c, d \text{ is } p^4 = p^{\#(f)}$$

So

$$|C(f)| = p^{\#(f)}$$

Cor. let X, C as above

Given G a perm gp on X

then

$$N(G, C) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|}$$

where $\#(f) = \#$ of cycles in the cycle factorization of f

Given $X = \{1, 2, \dots, n\}$

Given a perm $f \in X$

We now associate with f a polynomial in n variables

$$z_1, z_2, \dots, z_n$$

Consider cycle factorization of f

For $1 \leq k \leq n$ let

$$e_k = \# \text{ } k\text{-cycles in this factorization}$$

So

$$\sum_{k=1}^n e_k k = n$$

Note $\sum_{k=1}^n e_k = \# \text{ of cycles in this factorization} = \#(f)$

Define

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

"the monomial of f "

ex $n=8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 5 & 7 & 3 & 6 & 4 & 2 \end{pmatrix}$$

$$f = [182] \circ [35] \circ [47] \circ [6]$$

k	1	2	3	4	5	6	7	8
e_k	1	2	1	0	0	0	0	0

$$\text{mon}(f) = z_1 z_2^2 z_3$$

Def Given $X = \{1, 2, \dots, n\}$

Given $G = \text{perm group on } X$

Define a polynomial P_G in n variables

$$z_1, z_2, \dots, z_n$$

by

$$P_G(z_1, z_2, \dots, z_n) = \frac{\sum_{f \in G} m_m(f)}{|G|}$$

"cycle
index"
of G

Thm Given $X = \{1, 2, \dots, n\}$

Given $p \geq 1$

Given dist colors $c_1, c_2, \dots, c_p$

let $C = \text{set of all colorings of } X \text{ with these colors}$

let $G = \text{perm group on } X$

Then

$$N(G, C) = P_G(p, p, \dots, p)$$

$p \nmid$ Recall

$$N(G, C) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|}$$

By def

$$P_G(z_1, z_2, \dots, z_n) = \frac{\sum_{f \in G} \text{mm}(f)}{|G|}$$

For $f \in G$ write

$$\text{mm}(f) = z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

setting $z_i = p$ for $i \in [n]$, this becomes

$$p^{e_1} p^{e_2} \dots p^{e_n}$$

$$= p^{e_1 + e_2 + \dots + e_n}$$

$$= p^{\#(f)}$$

So

$$P_G(p, p, \dots, p) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|}$$

$$= N(G, C)$$

□

