Math 846

Lecture 38

Recall our DRG $\Gamma = (X, R)$ with $D \geq 2$

Assume Γ is Q poly wrt $\{E_i\}_{i=0}^{D}$

Fix $x \in X$ and write $T = T(x)$

Recall

$$V_{ij} = (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_j V)$$

for $-1 \leq i, j \leq D$

LEM 53 Fn $0 \le i, j \le D$

$$\dim \tilde{V}_{ij} = \dim V_{ij} - \dim V_{i-1,j} - \dim V_{i,j-1} + \dim V_{i-1,j-1}$$

pf We have

$$V_{ij} = \tilde{V}_{ij} \underset{\substack{\uparrow \\ ds}}{+} (V_{i-1,j} + V_{i,j-1})$$

So

$$\dim V_{ij} = \dim \tilde{V}_{ij} + \dim(V_{i-1,j} + V_{i,j-1})$$

By linear algebra, for any subspaces Y, Z of any f.d vector space,

$$\dim(Y+Z) + \dim(Y \cap Z) = \dim Y + \dim Z$$

So

$$\dim(V_{i-1,j} + V_{i,j-1}) = \dim V_{i-1,j} + \dim V_{i,j-1} - \dim(V_{i-1,j} \cap V_{i,j-1})$$

obs

$$V_{i-1,j} \cap V_{i,j-1} = V_{i-1,j-1}$$

Result follows. $\square$

Thm 54 For $0 \le r, s \le D$ Lec 38 3

$$V_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij} \qquad \text{(dir sum)}$$

pf Show

$$V_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij} \qquad (*)$$

We show $(*)$ by induction on $r+s$

$(*)$ holds for $r+s=0$ since $\tilde{V}_{00} = V_{00}$

by constr.

Assume $r+s > 0$.

By constr

$$V_{rs} = \tilde{V}_{rs} + V_{r,s-1} + V_{r-1,s}$$

By ind

$$V_{r,s-1} = \sum_{i=0}^{r} \sum_{j=0}^{s-1} \tilde{V}_{ij}$$

$$V_{r-1,s} = \sum_{i=0}^{r-1} \sum_{j=0}^{s} \tilde{V}_{ij}$$

Combining these equations we get $(*)$.

Show sum $(*)$ is direct.

Using Lem 53,

$$\dim V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \dim \widetilde{V}_{ij}$$

Therefore the sum $(*)$ is direct. $\square$

COR 55 We have

$$V = \sum_{i=0}^{\rho} \sum_{j=0}^{\rho} \widetilde{V}_{ij} \qquad (\text{dir sum})$$

pf set $r=s=\rho$ in Thm 54.

DEF 56 We call the sum in COR 55 the <u>split composition of V wrt x</u>

Caution: the sum is not orthogonal in general.

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OpenPROBLEM 57 What is the combinatorial meaning of the split decomp?

For $0 \le i, j \le D$ find a basis for $\tilde{V}_{ij}$ and find the action of A, A^* on this basis.

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We will return to the split decomp shortly

More about the irred T-modules.

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\Gamma = (X, R)$ is a DRG with diam $D \geq 2$

We do not assume Γ is Q-poly.

Fix $x \in X$ write $T = T(x)$

Fix an irred T-module W

Recall

$r =$ endpt of W

$d =$ diam of W

$t =$ dual endpt of W

$d^* =$ dual diam

Recall

W is thin whenever $\dim E_i^* W \leq 1$ for $0 \leq i \leq D$.

-- dual thin -- -- -- $\dim E_i W \leq 1$ --

LEM 58 With above notation,

(i) $E_i^* A E_j^* W \neq 0$ if $|i-j| = 1$ $\quad (r \le i,j \le r+d)$

(ii) Assume W is thin. Then

$$E_r^*W + E_{r+1}^*W + \cdots + E_{r+i}^*W = E_r^*W + AE_r^*W + \cdots + A^iE_r^*W$$

$$(0 \le i \le d)$$

(iii) Assume W is thin. Then $W = ME_r^*W$

(iv) Assume W is thin. Then

$$E_j E_r^* W = E_j W \qquad (0 \le j \le D)$$

Moreover W is dual thin.

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pf Suppose $\exists\ i, j\ (r \le i, j \le r+d)$ st

$$E_i^* A E_j^* W = 0 \quad \text{and} \quad |i-j| = 1.$$

If $i - j = 1$ then

$$E_r^* W + \cdots + E_j^* W$$

is a nonO proper subspace of W that is closed under A and M^*. But A, M^* generate T so this contradicts the irreducibility of W.

If $j - i = 1$ then

$$E_j^* W + \cdots + E_{r+d}^* W$$

is a nonO proper subspace of W that is closed under A and M^* contradicting the irreducibility of W.

(ii) By (i) and since

$$E_i^* A E_j^* = 0 \text{ if } |i-j| > 1 \quad (0 \le i, j \le D)$$

(iii) $\supseteq$: Since W is M-invariant

$\subseteq$: Set $i = d$ in (ii) and

use

$$W = \sum_{\ell=0}^{d} E_{r+\ell}^* W$$

(iv) For $0 \le j \le D$,

$$E_j W = E_j M E_r^* W$$

$$= E_j E_r^* W \qquad E_j M = \text{Span}(E_j)$$

Now

$$\dim E_j W \le \dim E_r^* W = 1 \qquad \square$$

We give the dual to Lem 38

LEM 39. With above notation, assume that Γ is Q-poly wrt $\{E_i\}_{i=0}^{D}$. Then

(i) $E_i A^* E_j W \neq 0$ if $|i-j|=1$ $\quad (t \le i, j \le t+d^*)$

(ii) Assume W is dual thin. Then

$$E_t W + \cdots + E_{t+i} W = E_t W + A^* E_t W + \cdots + (A^*)^i E_t W \quad (0 \le i \le d^*)$$

(iii) Assume W is dual thin. Then $W = M^* E_t W$

(iv) Assume W is dual thin. Then

$$E_j^* E_t W = E_j^* W \qquad (0 \le j \le D)$$

Moreover W is thin.

pf Similar to the proof of Lem 38. $\square$

Next we get some inequalities on r, t. We need a lemma:

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LEM 60 With above notation the following hold for $0 \le h \le D$

(i) For $0 \ne w \in E_h^* V$,

$$\left| \{ i \mid 0 \le i \le D, \quad E_i w = 0 \} \right| \le 2h$$

(ii) Assume Γ is Q-poly wrt $\{E_i\}_{i=0}^{D}$. Then for $0 \ne w \in E_h V$,

$$\left| \{ i \mid 0 \le i \le D, \quad E_i^* w = 0 \} \right| \le 2h$$

pf (i) Suppose not. Then $\exists$ subset

$$\Omega \subseteq \{0, 1, 2, \ldots, D\}, \qquad |\Omega| = 2h+1$$

st

$$E_i w = 0 \qquad \forall i \in \Omega$$

By constr $E_h^* w = w$.

For $i \in \Omega$,

$$
\begin{aligned}
0 &= E_h^* E_i w \\
&= E_h^* E_i E_h^* w \\
&= E_h^* \left(|X|^{-1} m_i \sum_{\ell=0}^{D} u_\ell(\theta_i) \, A_\ell \right) E_h^* w
\end{aligned}
$$

$$E_h^* A_\ell E_h^* = 0 \quad \text{for } \ell > 2h$$

So

$$0 = \sum_{\ell=0}^{2h} u_\ell(\theta_i) \, E_h^* A_\ell E_h^* w$$

Letting i range over Ω we get a system of $2h+1$ homogeneous linear equations in the unknowns

$$E_h^* A_\ell E_h^* w \qquad 0 \le \ell \le 2h$$

The coefficient matrix is essentially Vandermonde

since the polynomial u_ℓ has degree ℓ

and the θ_i are distinct.

Therefore the coefficient matrix is invertible, forcing

$$E_h^* A_\ell E_h^* w = 0 \qquad 0 \le \ell \le 2h$$

This is a contradiction since

$$E_h^* A_0 E_h^* w = E_h^* w = w \neq 0.$$

(ii) Similar to the proof of (i) $\square$

COR 61 With the above notation, let W denote an irred T-module with endpt r, dual endpt t, diam d, dual diam d^*. Then

(i) $2r + d^* \geq D$

(ii) Assume that Γ is Q-poly w.r.t $\{E_i\}_{i=0}^{D}$

then $2t + d \geq D$

pf (i) Pick $0 \neq w \in E_r^* W$. By Lem 50 (i),

$$2r \geq \left| \{ i \mid 0 \leq i \leq D, \ E_i w = 0 \} \right|$$

$$\geq \left| \{ i \mid 0 \leq i \leq D, \ E_i W = 0 \} \right|$$

$$= D - d^*$$

(ii) Similar to the proof of (i) $\square$