

Combinatorics 475

Lecture 37

14.2 Cont

Recall Burnside

Given:

 $X$  a nonempty finite set $G$  a perm gp on  $X$  $C$  a nonempty set of colourings of  $X$  that is  $G$ -closed  
fixed pts of  $f$ 

Then

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

↑  
#  $\sim^G$ -equivalence  
in  $C$

Ex Consider multiset

$$S = \{a, a, a, b, b, c, c, d, d\}$$

Fix integer  $n \geq 1$ .

Recall # of  $n$ -perms of  $S$  is  $4^n$ .

How many  $n$ -perms of  $S$  if we declare each

$a_1, a_2, \dots, a_n$  and its mirror image  $a_n, \dots, a_2, a_1$

to be equivalent?

[so for  $n=6$   $abbccdd$  equiv  $ddccbb$ ]

Sol Define

$$X = \{1, 2, \dots, n\}$$

View  $a, b, c, d$  as colors

$C =$  set of all colorings of  $X$  with  $a, b, c, d$

$$|C| = 4^n$$

Define

$$G = \{I, \tau\}$$

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$$

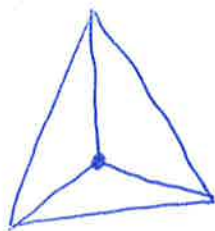
$$\tau^2 = I$$

$G$  is a perm gp on  $X$



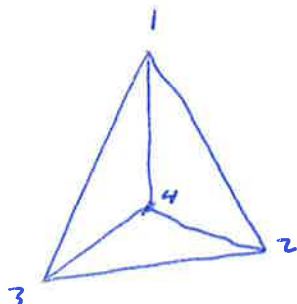
$E_x$ 

Tetrahedron



"in 3 dimensions"

"top view"

 $X = \text{set of vertices}$ Label  $X$ the gp  $G$  of symmetries of the tetrahedron

| types of symmetries in $G$ | #  |
|----------------------------|----|
| $120^\circ$ clockwise rot  | 4  |
| $120^\circ$ c.c. rot       | 4  |
| $180^\circ$ rot            | 3  |
| identity $I$               | 1  |
|                            | 12 |

We allow only "physically possible" symmetries

$$|G| = 12$$

$G$  is often called the alternating group  $A_4$

For the above tetrahedron

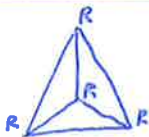
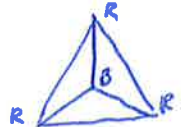
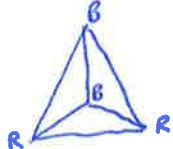
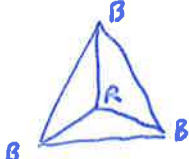
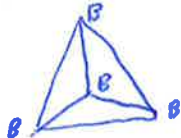
let

$C =$  set of colorings of  $X$  with colors  $R, B$

So  $|C| = 2^4 = 16$

Find  $N(G, C)$

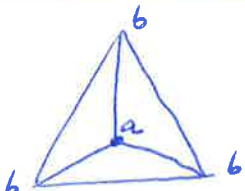
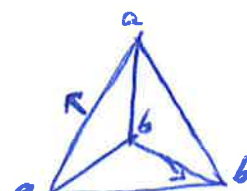
Sol 1: describe each  $G$  equiv class

| #B | desc                                                                                | #colorings |
|----|-------------------------------------------------------------------------------------|------------|
| 0  |   | 1          |
| 1  |  | 4          |
| 2  |  | 6          |
| 3  |  | 4          |
| 4  |  | 1          |

Each row gives a  $\sim^G$  equiv class

$$N(G, C) = \# \text{rows} = 5$$

Sol 2: Use Burnside

| $f \in G$                    | $C(f)$ desc                                                                                             | $ C(f) $  |
|------------------------------|---------------------------------------------------------------------------------------------------------|-----------|
| $120^\circ$ clockwise<br>rot |  $a, b \in \{R, B\}$   | $2^2 = 4$ |
| $120^\circ$ ccw<br>rot       | "                                                                                                       | "         |
| $180^\circ$ rot              |  $a, b \in \{R, B\}$ | $2^2 = 4$ |
| I                            | C                                                                                                       | $2^3 = 8$ |

$$\begin{aligned}
 N(G, C) &= \frac{\sum_{f \in G} |C(f)|}{|G|} \\
 &= \frac{4 \times 4 + 4 \times 4 + 3 \times 4 + 1 \times 8}{12} \\
 &= \frac{60}{12} \\
 &= 5
 \end{aligned}$$

□

Ex For the above tetrahedron

let

$C = \text{set of colorings of } X \text{ with } R, W, B$

So  $|C| = 3^4 = 81$

Find  $N(G, C)$

Sol 1 describe each  $\sim^G$  equiv class

For a coloring  $c \in C$

define

$r = \# \text{ vertices colored } R$

$w = \dots W$

$b = \dots B$

upto  $\sim^G$   $c$  is det by  $r, w, b$

obs  $r + w + b = 4$

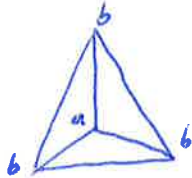
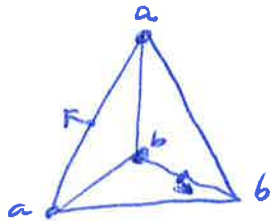
$$r \geq 0, w \geq 0, b \geq 0$$

#sols for  $r, w, b$  is

$$\binom{4+3-1}{3-1} = \binom{6}{2} = 15$$

$$N(G, C) = 15$$

## Sol 2 Use Burnside

| $f \in G$               | $C(f)$ desc                                                                                               | $ C(f) $  |
|-------------------------|-----------------------------------------------------------------------------------------------------------|-----------|
| $120^\circ$ cl<br>rot   |  $a, b \in \{R, W, B\}$  | $3^2 = 9$ |
| $120^\circ$ c.c.<br>rot | ..                                                                                                        | ..        |
| $180^\circ$ rot         |  $a, b \in \{R, W, B\}$ | $3^2 = 9$ |
| I                       | C                                                                                                         | $3^4$     |

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

$$= \frac{4 \times 3^2 + 4 \times 3^2 + 3 \times 3^2 + 1 \times 3^4}{12}$$

$$= 15$$

Ex For above tetrahedron

Given integer  $p$

Given  $p$  dist colors

Let  $C =$  set of colorings of  $X$  with these colors

$$\text{so } |C| = p^4$$

Find  $N(G, C)$

Use Sol 2: By Burnside

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

$$= \frac{4 \times p^2 + 4 \times p^2 + 3 \times p^2 + 1 \times p^4}{12}$$

$$= \frac{p^2(p^2 + 11)}{12}$$

Sol 1: gets complicated

□

Ex For tetrahedron

Now take  $G = S_4$  = set of all perms of  $X$

$$|G| = 4! = 24$$

Given  $p \geq 1$

Given  $p$  dist colors  $c_1, c_2, \dots, c_p$

$C$  = set of colorings of  $X$  with these colors

Find  $N(G, C)$

Sol 1 Desc each equiv class for  $\sim^G$

For a coloring  $c \in C$

For  $i=1, \dots, p$  let

$n_i = \# \text{ vertices colored } c_i$

$c$  is det up to  $\sim^G$  by

the sequence

$$n_1, n_2, \dots, n_p$$

Obs

$$n_1 + n_2 + \dots + n_p = 4$$

$$n_i \geq 0$$

126p

# sols are

$$\binom{4+p-1}{p-1} = \binom{p+3}{4}$$

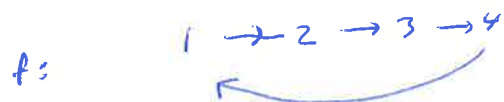
$$N(G, C) = \binom{p+3}{4}$$

Sol 2 Use Burnside

Describe the elements of  $G$

ex  $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

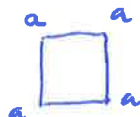
$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$



cycle type



| cycle type                                                                          | # els of $G$       |
|-------------------------------------------------------------------------------------|--------------------|
|  | $3! = 6$           |
|  | $4 \times 2 = 8$   |
|  | 3                  |
|  | $\binom{4}{2} = 6$ |
|  | 1                  |

| $f \in G$                                                                           | $C(f)$ desc                                                                        | $ C(f) $         |
|-------------------------------------------------------------------------------------|------------------------------------------------------------------------------------|------------------|
|    |   | $a$ arb          |
|    |   | $a, b$ arb       |
|    |   | $a, b$ arb       |
|    |   | $a, b, c$ arb    |
|  |  | $a, b, c, d$ arb |

$$\begin{aligned}
 N(G, C) &= \frac{\sum_{f \in G} |C(f)|}{|G|} \\
 &= \frac{6p + 8p^2 + 3p^2 + 6p^3 + p^4}{24} \\
 &= \frac{6p + 11p^2 + 6p^3 + p^4}{24} \\
 &= \frac{(p+3)(p+2)(p+1)p}{24} \\
 &= \binom{p+3}{4}
 \end{aligned}$$

□

