Math 846

Lecture 37

Until further notice, assume $\Gamma$ is $Q$-poly wrt $\{E_i\}_{i=0}^{D}$. Let the scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ be from the relations TD1, TD2.

Thm 44 With above notation,

(i) For $2 \leq i \leq D$

$$g_i^+ L^2 F + LFL + g_i^- FL^2 - \gamma L^2 = 0$$

on $E_i^* V$, where

$$g_i^+ = \frac{\theta_{i-2}^* - (\beta+1)\theta_{i-1}^* + \beta\theta_i^*}{\theta_{i-2}^* - \theta_i^*}$$

$$g_i^- = \frac{-\beta\theta_{i-2}^* + (\beta+1)\theta_{i-1}^* - \theta_i^*}{\theta_{i-2}^* - \theta_i^*}.$$

(ii) For $1 \le i \le D-1$

$$\left[ F, \; LR - h_i RL \right] = 0$$

on $E_i^* V$, where

$$h_i = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*}.$$

Moreover $[F, RL] = 0$ on $E_D^* V$.

(iii) For $1 \le i \le D$,

$$e_i^+ L^2 R + (\beta+2) LRL + e_i^- RL^2$$

$$+ LF^2 - \beta FLF + F^2 L - \gamma(LF+FL) - \delta L$$

$$= 0$$

on $E_i^* V$, where

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta+2)\theta_i^* + (\beta+1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \qquad \text{if } 1 \le i \le D-1$$

$$e_i^- = \frac{-(\beta+1)\theta_{i-2}^* + (\beta+2)\theta_{i-1}^* - \theta_i^*}{\theta_{i-1}^* - \theta_i^*} \qquad \text{if } 2 \le i \le D$$

and $e_D^+$, $e_1^-$ are indeterminates

Pf Recall TD1:

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*]$$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2)$$
$$- \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

Call the above expression $\Psi$ so

$$0 = \Psi$$

(i) Obs $\quad 0 = E^*_{i-2} \Psi E^*_i$

Show that on $E^*_i V$,

$$E^*_{i-2} \Psi E^*_i = g_i^+ L^2 F + LFL + g_i^- FL^2 - \gamma L^2$$

To show this, for each term in $\Psi$ multiply on the left by $E^*_{i-2}$ and the right by $E^*_i$, and simplify.

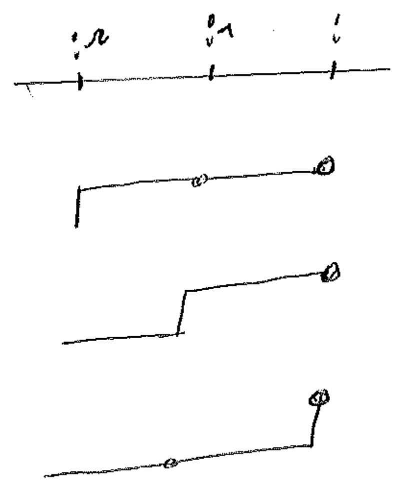
For example

$$E^*_{i-2} A^3 A^* E^*_i = \theta^*_i E^*_{i-2} A^3 E^*_i$$

$$E^*_{i-2} A^3 E^*_i = E^*_{i-2} A \left( \sum_{r=0}^{D} E^*_r \right) A \left( \sum_{s=0}^{D} E^*_s \right) A E^*_i$$

$$= \sum_{r=0}^{D} \sum_{s=0}^{D} E^*_{i-2} A E^*_r A E^*_s A E^*_i$$

$$= \sum_{\substack{0 \le r, s \le D \\ |i-2-r| \le 1 \\ |r-s| \le 1 \\ |s-i| \le 1}} E^*_{i-2} A E^*_r A E^*_s A E^*_i$$




$$= E^*_{i-2} A E^*_{i-2} A E^*_{i-1} A E^*_i$$

$$+ E^*_{i-2} A E^*_{i-1} A E^*_{i-1} A E^*_i$$

$$+ E^*_{i-2} A E^*_{i-1} A E^*_i A E^*_i$$

$$= \left( FL^2 + LFL + L^2 F \right) E^*_i .$$

Similarly

$$E^*_{i-2} A^* A^3 E^*_i = \theta^*_{i-2} \left( FL^2 + LFL + L^2F \right) E^*_i$$

$$E^*_{i-2} A^2 A^* A E^*_i = \left( \theta^*_{i-1} ( FL^2 + LFL ) + \theta^*_i L^2 F \right) E^*_i$$

$$E^*_{i-2} A A^* A^2 E^*_i = \left( \theta^*_{i-2} FL^2 + \theta^*_{i-1} \left( LFL + L^2F \right) \right) E^*_i$$

$$E^*_{i-2} \left( A^2 A^* - A^* A^2 \right) E^*_i = \left( \theta^*_i - \theta^*_{i-2} \right) L^2 E^*_i$$

$$E^*_{i-2} \left( A A^* - A^* A \right) E^*_i = 0$$

Result follows

(ii) Evaluate

$$0 = E^*_i \Psi E^*_i$$

as in (i)

(iii) Evaluate

$$0 = E^*_{i-1} \Psi E^*_i$$

as in (i) $\square$

LEM 45 With reference to Thm 44

$$e_i^+ = \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+1}^*} \qquad 1 \le i \le D-2$$

$$e_i^- = \frac{\theta_{i-1}^* - \theta_{i-3}^*}{\theta_{i-1}^* - \theta_i^*} \qquad 3 \le i \le D$$

$$g_i^+ = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-2}^*} \qquad 2 \le i \le D-1$$

$$g_i^- = \frac{\theta_{i-2}^* - \theta_{i-3}^*}{\theta_{i-2}^* - \theta_i^*} \qquad 3 \le i \le D$$

In particular $e_i^\pm$, $g_i^\pm$ are non 0 for the range of $i$ given above.

pf use

$$\beta+1 = \frac{\theta_{j-2}^* - \theta_{j+1}^*}{\theta_{j-1}^* - \theta_j^*} \qquad 2 \le j \le D-1$$

$\square$

Note 46 With reference to Thm 44

assume $\Gamma$ is bipartite, then $F = 0$

by construction and $\gamma = 0$ by Thm 19 (Lec 24)

So (i) yields no info.

and (ii) yields no info

(iii) implies for $1 \leq i \leq D$

$$e_i^+ L^2 R + (\beta+2) LRL + e_i^- RL^2 - \delta L = 0$$

on $E_i^* V$

No longer assume that $\Gamma$ is Q-poly.

Open Problem 47. Assume that the DRG $\Gamma = (X, R)$ is bipartite with $D \geq 2$. Fix $x \in X$ and write $T = T(x)$

Assume that for $1 \leq i \leq D$,

$$L^2R, \quad LRL, \quad RL^2, \quad L$$

are lin dep on $E_i^* V$

(coef may vary with $i$)

Then is $\Gamma$ Q-polynomial?

If not, then what other condition is needed?

<u>Note</u> the partially ordered sets that satisfy the above condition are called <u>uniform</u>

Excercise VII  Assume that the DRG $\Gamma = (X, R)$ is bipartite with $D \geq 2$.

Assume the ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite Q poly.

Fix $x \in X$ and write $T = T(x)$.

- Show that for $1 \leq i \leq D$ the matrices

$$LR, \quad RL, \quad I$$

are lin dep on $E_i^* V$ (coef may depend on $i$)

- Find the above coefficients.

- Show that for all $y, z \in X$ and $0 \leq r, s, t \leq D$ the triple intersection number

$$|\Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z)|$$

depends only on $r, s, t$ and

$$\partial(x,y), \quad \partial(x,z), \quad \partial(y,z).$$

Next topic: the split decomposition. Lec 37

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$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, Given DRG $\Gamma = (X, \mathcal{R})$ diam $D \geq 2$

Assume $\Gamma$ is Q-poly wrt $\{E_i\}_{i=0}^{D}$.

Fix $x \in X$ and write $T = T(x)$. Recall the orthogonal

direct sums

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V, \qquad V = E_0 V + E_1 V + \cdots + E_D V$$

$V$ = standard module.

DEF 48 For $-1 \leq i, j \leq D$ define

$$V_{ij} = (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_j V)$$

We Interp $V_{ij} = 0$ if $i = -1$ or $j = -1$.

LEM 49 For $0 \leq i, j \leq D$ and $v \in V$ TFAE:

(i) $v \in V_{ij}$

(ii) $E_h^* v = 0$ for $i < h \leq D$ and $E_h v = 0$ for $j < h \leq D$

Pf By Def 48. $\square$

LEM 50 For $0 \le i, j \le D$,

$$V_{i+1,j} \subseteq V_{ij}, \qquad V_{i,j-1} \subseteq V_{ij}$$

pf By Def 48. $\square$

LEM 51 We have.

(i) $V_{i0} = E_0^* V + E_1^* V + \cdots + E_i^* V \qquad (0 \le i \le D)$

(ii) $V_{Dj} = E_0 V + E_1 V + \cdots + E_j V \qquad (0 \le j \le D)$

(iii) $V_{D0} = V$

pf By Def 48 $\square$

It turns out that

$$V_{ij} = 0 \quad \text{if} \quad i+j < D \qquad (0 \le i, j \le D)$$

Def 52 For $0 \le i, j \le D$ observe that

$$V_{i+1,j} + V_{i,j+1} \subseteq V_{ij}$$

Let $\tilde{V}_{ij}$ denote the orthogonal complement

of $V_{i+1,j} + V_{i,j+1}$ in $V_{ij}$.

So

$$V_{ij} = \tilde{V}_{ij} + \left( V_{i+1,j} + V_{i,j+1} \right)$$

↑ orthog direct sum.

For notational convenience define

$$\tilde{V}_{ij} = 0 \quad \text{if} \quad i \notin \{0, 1, \ldots, D\} \text{ or } j \notin \{0, 1, \ldots, D\}$$

Our next goal is to show

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij} \qquad \text{(direct sum)}$$