

Combinatorics 475

Lecture 36

14.2 Burnside's Theorem

Let

 $X =$ nonempty finite set

$$X = \{1, 2, \dots, n\}$$

 $G =$ a permutation group on X

Pick some colors and consider the set of all colorings of X with those colors.

We saw that G induces a perm gp on that set

Let $C =$ a nonempty subset of the above set of colorings of X

Call C G -closed whenever

$$f \cdot c \in C \quad \text{for all } f \in G \text{ and } c \in C$$

In this case G induces perm gp on C

REX For each row in table I, the given set of colorings is G -closed.

From now on assume C is G -closed.

Recall that for colorings $c_1, c_2 \in C$

$$c_1 \stackrel{G}{\sim} c_2 \quad \text{means}$$

" G -equivalent"

$$\exists f \in G \text{ s.t. } f * c_1 = c_2$$

$\stackrel{G}{\sim}$ is an equivalence relation on C

C = disjoint union of $\stackrel{G}{\sim}$ equivalence classes

Define

$$N(G, C) = \# \text{ equivalence classes of } \stackrel{G}{\sim} \text{ on } C$$

$$= \# \text{ of mutually } G\text{-inequivalent colorings in } C$$

REX If we take

$$C = \text{all 16 colorings of } X$$

$$\text{then } N(G, C) = 6.$$

Since there is 1 $\stackrel{G}{\sim}$ equiv class for each of the 6 rows of table I.

Next goal: Find formula for $N(G, C)$

$\forall c \in C$ and $f \in G$

we say

f stabilizes c

or

f fixes c

whenever

$$f \cdot c = c$$

$\forall c \in C$ define

$$G(c) = \{ f \in G \mid f \cdot c = c \}$$

"the stabilizer of c in G "

LEM

Given $c \in C$, then $G(c)$ is a permutation gp on C pf • Given $f, g \in G(c)$ show $f \circ g \in G(c)$:We have $f * c = c$, $g * c = c$

Obs

$$\begin{aligned}
 (f \circ g) * c &= f * (g * c) \\
 &= f * c \\
 &= c
 \end{aligned}$$

• Show $I \in G(c)$:

$$I * c = c \quad \checkmark$$

□

LEM Given $c \in C$.

Then for all $f, g \in G$ the following are equivalent:

(i) $f * c = g * c$

(ii) $f^{-1} \circ g \in G(c)$

pf

$$f * c = g * c$$

$\Leftrightarrow$

$$f^{-1} * (f * c) = f^{-1} * (g * c)$$

"

"

$$(f^{-1} \circ f) * c$$

$$(f^{-1} \circ g) * c$$

"

$$I * c$$

"

$$c$$

$\Leftrightarrow$

$$f^{-1} \circ g \in G(c)$$

□

Given $c \in C$

Consider the equivalence class $\sim^G$ containing c
this is

$$\{ f * c \mid f \in G \}$$

thm For $c \in C$

$$|\{ f * c \mid f \in G \}| = \frac{|G|}{|G(c)|}$$

pf Abbr

$$Y = \{ f * c \mid f \in G \}$$

For $y \in Y$ define

$$G^{(y)} = \{ f \in G \mid f * c = y \}$$

$\{ G^{(y)} \}_{y \in Y}$ partition G

so

$$|G| = \sum_{y \in Y} |G^{(y)}|$$

For $y \in Y$ show

$$|G^{(y)}| = |G(c)|$$

For $f \in G^{(y)}$, the map

$$\begin{aligned} G(c) &\rightarrow G^{(y)} \\ h &\rightarrow f \circ h \end{aligned}$$

is a bijection (by prev LEM)

So

$$|G(c)| = |G^{(y)}|$$

Now

$$|G| = \sum_{y \in Y} |G^{(y)}|$$

" $|G(c)|$

$$= |Y| |G(c)|$$

So

$$|Y| = \frac{|G|}{|G(c)|}$$

□

Def For $f \in G$ define

$$C(f) = \{c \in C \mid f * c = c\}$$

"set of colorings in C that are fixed by f "

So for $f \in G$ and $c \in C$

$$c \in C(f) \iff f * c = c \iff f \in G(c)$$

thm (Burnside)

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

of equivalence classes of $\sim_G$ on C

pf let

$S =$ set of ordered pairs (f, c) such that
 $f \in G$ and $c \in C$ and $f * c = c$

We compute $|S|$ in two ways

$$\text{I} \quad |S| = \sum_{f \in G} |C(f)|$$

$$\text{II} \quad |S| = \sum_{c \in C} |G(c)|$$

let $C_1, C_2, \dots, C_N$ denote the equiv classes
of $\sim_G$ on C

$$\begin{aligned}
 |S| &= \sum_{c \in C} |G(c)| \\
 &= \sum_{i=1}^N \sum_{c \in C_i} |G(c)| \\
 &\quad \quad \quad \underbrace{\quad}_{\substack{\text{by prev thm} \\ |G| \\ |C_i|}} \\
 &= \sum_{i=1}^N |C_i| \frac{|G|}{|C_i|} \\
 &= \sum_{i=1}^N |G| \\
 &= N |G|
 \end{aligned}$$

So

$$\begin{aligned}
 N &= \frac{|S|}{|G|} \\
 &= \frac{\sum_{f \in G} |C(f)|}{|G|}
 \end{aligned}$$

□

REX Take

C = set of all 16 colorings of X using colors R, B

G = dihedral gp of order 8

We saw $N(G, C) = 6$

Let us verify this using Burnside.

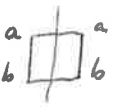
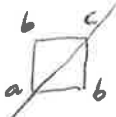
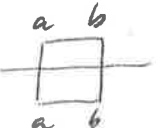
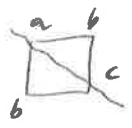
By Burnside

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

$$|G| = 8$$

$$G = \{1, p, p^2, p^3, r, pr, p^2r, p^3r\}$$

For all $f \in G$ find $|C(f)|$

$f \in G$	$C(f)$ desc	$ C(f) $
1	C	$2^4 = 16$
r	 $a \in \{R, B\}$	2
p^2	 $a, b \in \{R, B\}$	$2^2 = 4$
p^3	 $a \in \{R, B\}$	2
r^2	 $a, b \in \{R, B\}$	$2^2 = 4$
$p r$	 $a, b, c \in \{R, B\}$	$2^3 = 8$
$p^2 r$	 $a, b \in \{R, B\}$	$2^2 = 4$
$p^3 r$	 $a, b, c \in \{R, B\}$	$2^3 = 8$
		48

$$N(G, C) = \frac{48}{8} = 6 \quad \checkmark$$

□

Ex X, G as above

Given integer $p \geq 1$

Given p distinct colors $c_1, \dots, c_p$

Let $C =$ set of colorings of X with these colors.

So $|C| = p^4$

Find $N(G, C)$

Sol prev ex is case $p=2$

In prev ex replace 2 by p

f	$ C(f) $
1	p^4
p	p
p^2	p^2
p^3	p
τ	p^2
$p\tau$	p^3
$p^2\tau$	p^2
$p^3\tau$	p^3
$p^4 + 2p^3 + 3p^2 + 2p$	

$$N(G, C) = \frac{p^4 + 2p^3 + 3p^2 + 2p}{8}$$

p	$N(G, C)$
1	1
2	6
3	21
4	55
5	126

