Math 846

Lecture 36

Next we consider the combinatorial consequences of the Q polynomial property.

thm 38 (Heather Lewis)

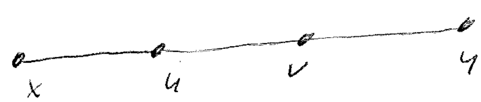
Given a Q-polynomial DRG $\Gamma = (X, R)$ with diameter $D \geq 3$. Assume that Γ is not a cycle. Then the intersection number $c_3 \geq 2$.

pf The valency $k \geq 3$.

We assume that $c_3 = 1$ and get a contradiction.

Pick $x, y \in X$ at $\partial(x, y) = 3$

$\exists$ unique path of length 3 from x to y:



By assumption Γ is Q-poly wrt some prim idempotent E

Write $E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$

By thm 37

$$\sum_{w\in\Gamma(x)\cap\Gamma_2(y)} E\hat{w} \;-\; \sum_{w\in\Gamma_2(x)\cap\Gamma(y)} E\hat{w} \;=\; p^3_{12}\,\frac{\theta_1^*-\theta_2^*}{\theta_0^*-\theta_3^*}\left(E\hat{x}-E\hat{y}\right)$$

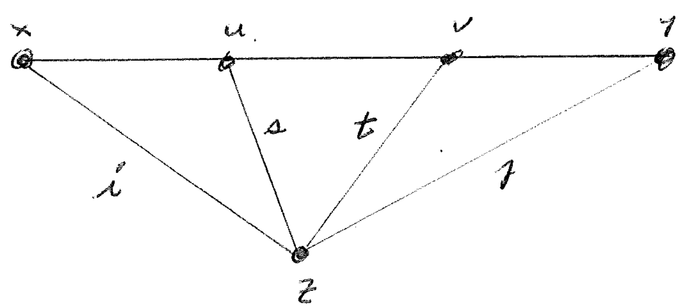
Obs $\quad \Gamma(x)\cap\Gamma_2(y)=\{u\}, \quad \Gamma_2(x)\cap\Gamma(y)=\{v\}, \quad p^3_{12}=1$

So

$$E\hat{u} - E\hat{v} = \frac{\theta_1^*-\theta_2^*}{\theta_0^*-\theta_3^*}\left(E\hat{x}-E\hat{y}\right) \qquad (*)$$

Pick any $z\in X$, and define

$$i = \partial(x,z)$$
$$j = \partial(y,z)$$
$$s = \partial(u,z)$$
$$t = \partial(v,z)$$



Find inner product of $E\hat{z}$ with $(*)$

Get

$$(\theta_s^* - \theta_t^*)(\theta_0^* - \theta_j^*) = (\theta_i^* - \theta_2^*)(\theta_i^* - \theta_j^*) \qquad (**)$$

<u>claim</u> $p_1^1 = 0$

pf claim Suppose $p_1^1 \neq 0$. We can choose z st

$i = 1, \quad s = 1$

then $t = 2, \quad j = 3$ by the triangle inequality

and $c_3 = 1$.

Now $(**)$ becomes

$$(\theta_1^* - \theta_2^*)(\theta_0^* - \theta_3^*) = (\theta_1^* - \theta_2^*)(\theta_1^* - \theta_3^*)$$

forcing $\theta_0^* = \theta_1^*$ cont.

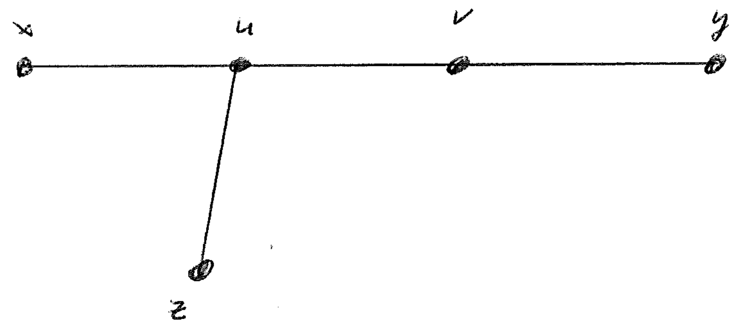
claim proved ✓

Since $k \geq 3$, we can choose

$$z \in \Gamma(u) \setminus \{x, v\}$$

By this and $p^1_{11} = 0$,

$i = 2$ $s = 1$, $t = 2$, $j \in \{2, 3\}$



Suppose $j = 2$. Then $(**)$ becomes

$$\underbrace{(\theta_1^* - \theta_2^*)}_{\neq 0}\underbrace{(\theta_0^* - \theta_3^*)}_{\neq 0} = (\theta_1^* - \theta_2^*)\underbrace{(\theta_2^* - \theta_2^*)}_{= 0}$$

cont

Suppose $j = 3$. Then $(**)$ becomes

$$(\theta_1^* - \theta_2^*)(\theta_0^* - \theta_3^*) = (\theta_1^* - \theta_2^*)(\theta_2^* - \theta_3^*)$$

forcing $\theta_0^* = \theta_2^*$ cont

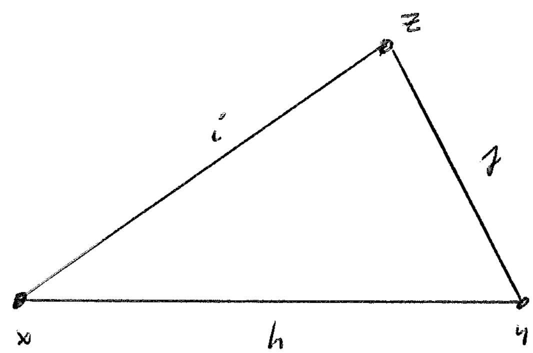
In both cases we get a contradiction, so $c_3 \geq 2$ $\square$

Given DRG $\Gamma = (X, R)$ diam $D \geq 1$

Given $x, y, z \in X$.

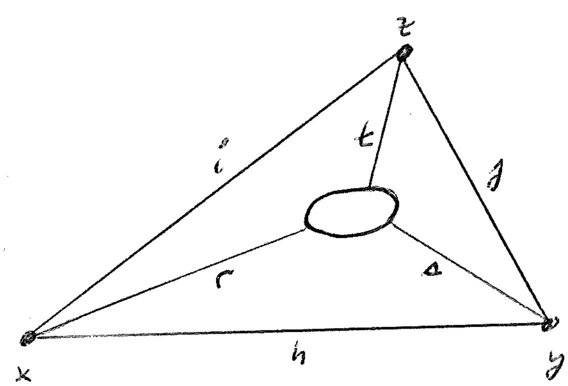
Define

$$h = \partial(x,y) \qquad i = \partial(x,z), \qquad j = \partial(y,z)$$



For $0 \leq r, s, t \leq D$ consider

$$\Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z)$$



In general

$$\left| \Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z) \right| \qquad (*)$$

is not a constant that depends only on h, i, j, r, s, t

We call $(*)$ a <u>triple intersection number</u>

for x, y, z.

Next we consider the constraints on the triple intersection numbers.

LEM 39 With above notation

$$\sum_{t=0}^{D} \left| \Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z) \right| = p^h_{rs}$$

pf Use

$$\Gamma_r(x) \cap \Gamma_s(y) = \bigcup_{t=0}^{D} \Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z) \qquad \text{disjoint union}$$

$\square$

COR 40 With above notation,

each of

$$p^h_{r s}, \quad p^i_{r t}, \quad p^j_{s t}$$

is at least

$$|\Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z)|$$

$\square$

pf By Lem 39 and symmetry.

Recall that for $0 \le a, b, c \le D$

$$E_b A^*_a E_c = 0 \quad \text{iff} \quad q^a_{bc} = 0$$

where $A^*_a = A^*_a(x)$

We now examine the entries of $E_b A^*_a E_c$

LEM 41 For $x, y, z \in X$ and $0 \le a, b, c \le D$, the (y,z)-entry of

$$E_b A_a^* E_c \qquad\qquad A_a^* = A_a^*(x)$$

is equal to

$$|X|^{-2} m_a m_b m_c$$

times

$$\sum_{r=0}^{D} \sum_{s=0}^{D} \sum_{t=0}^{D} \left| \Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z) \right| u_r(\theta_a) u_s(\theta_b) u_t(\theta_c)$$

pf Recall

$$E_a = |X|^{-1} m_a \sum_{r=0}^{D} u_r(\theta_a) A_r$$

Observe

$$(E_b A_a^* E_c)_{y,z}$$

$$= \sum_{w \in X} (E_b)_{yw} (A_a^*)_{ww} (E_c)_{wz}$$

$$= |X| \sum_{w \in X} (E_b)_{yw} (E_a)_{xw} (E_c)_{zw}$$

$$= |X| \sum_{w \in X} (E_a)_{xw} (E_b)_{yw} (E_c)_{zw}$$

$$= |X| \sum_{r=0}^{D} \sum_{s=0}^{D} \sum_{t=0}^{D} \sum_{w \in \Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z)} (E_a)_{xw} (E_b)_{yw} (E_c)_{zw}.$$

For $0 \le r,s,t \le D$ and $w \in \Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z)$

$$(E_a)_{xw} = |X|^{-1} m_a u_r(\theta_a),$$

$$(E_b)_{yw} = |X|^{-1} m_b u_s(\theta_b),$$

$$(E_c)_{zw} = |X|^{-1} m_c u_t(\theta_c).$$

Result follows. $\square$

Thm 42 For $0 \le a, b, c \le D$ TFAE:

(i) $q^a_{bc} = 0$

(ii) For $x, y, z \in X$

$$0 = \sum_{r=0}^{D} \sum_{s=0}^{D} \sum_{t=0}^{D} |\Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z)| \, u_r(\theta_a) u_s(\theta_b) u_t(\theta_c).$$

pf Use Lem 41. □

Suppose $\Gamma = (X, R)$ is Q-polynomial.
We have many vanishing krein parameters,
and hence many constraints on the triple
intersection numbers

— o —

The equations in Thm 42 involve a triple sum.
In the next result, the balanced set condition is used to get
similar equations that involve a single sum.

Thm 43 Assume that our DRG $\Gamma = (X, \mathcal{R})$ is Q-polynomial with respect to the prim idempotent

Lec 36 11

$$E = |X|^{-1} \sum_{t=0}^{D} \theta_t^* A_t.$$

Given $x, y, z \in X$ $(x \neq y)$ and write

$$h = \partial(x,y), \quad i = \partial(x,z), \quad j = \partial(y,z).$$

Then for $0 \le r, s \le D$

$$\sum_{t=0}^{D} |\Gamma_r(x) \cap \Gamma_s(y) \cap \Gamma_t(z)| \, \theta_t^* - \sum_{t=0}^{D} |\Gamma_s(x) \cap \Gamma_r(y) \cap \Gamma_t(z)| \, \theta_t^*$$

$$= p^h_{rs} \frac{\theta_r^* - \theta_s^*}{\theta_0^* - \theta_h^*} (\theta_i^* - \theta_j^*).$$

pf By thm 37

$$\sum_{w \in \Gamma_r(x) \cap \Gamma_s(y)} E\hat{w} - \sum_{w \in \Gamma_s(x) \cap \Gamma_r(y)} E\hat{w} = p^h_{rs} \frac{\theta_r^* - \theta_s^*}{\theta_0^* - \theta_h^*} (E\hat{x} - E\hat{y})$$

Take the inner product of each side with $E\hat{z}$ and simplify. $\square$

We now consider the combinatorial meaning of the Q-poly property from another point of view. Let $D \geq 2$.

Fix $x \in X$ and write $T = T(x)$, etc

Define

$$R = \sum_{i=0}^{D-1} E^*_{i+1} A E^*_i \qquad \text{"raise"}$$

$$F = \sum_{i=0}^{D} E^*_i A E^*_i \qquad \text{"flat"}$$

$$L = \sum_{i=1}^{D} E^*_{i-1} A E^*_i \qquad \text{"lower"}$$

Note that $R^t = L$ and $F^t = F$ and

$$A = R + F + L$$

Observe

$$R E^*_i V \subseteq E^*_{i+1} V \qquad 0 \leq i \leq D$$

$$F E^*_i V \subseteq E^*_i V \qquad 0 \leq i \leq D$$

$$L E^*_i V \subseteq E^*_{i-1} V \qquad 0 \leq i \leq D$$

where $E^*_{-1} = 0$, $E^*_{D+1} = 0$