

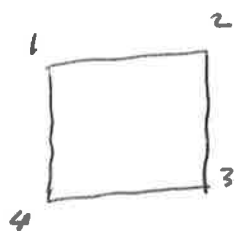
Combinatorics 475

Lecture 35

14.1 Cont.

Let $X =$ nonempty finite setSay $X = \{1, 2, \dots, n\}$ Let $G =$ permutation group on X

Running example (REX): X is set of vertices for the regular 4-gon



$G =$ the group of symmetries
 $=$ dihedral group of order 8

define

 $p:$ 90° clockwise rotation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

 $r:$ reflection about

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$G = \{I, p, p^2, p^3, r, pr, p^2r, p^3r\}$$

Def: A coloring of X is an assignment
of a color to each element of X
(distinct elements of X might get the same color)

REX Using colors Red (R) and Blue (B) there are
 $2^4 = 16$ possible colorings. They are:

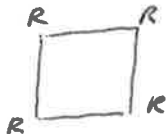
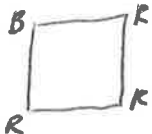
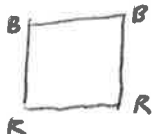
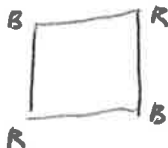
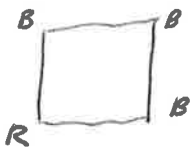
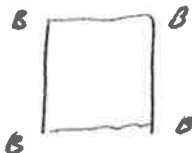
# B	desc	#colorings
0		1
1	 + cyclic perms	4
2	 ...	4
2	 ...	2
3	 ...	4
4		1

Table 1

View each coloring alone as a function

$$\begin{aligned} X &\longrightarrow \{R, B\} \\ i &\longrightarrow \text{color assigned to } i \end{aligned}$$

Given $f \in G$

Given a coloring c of X

Permute X to get another coloring of X :

$$\begin{array}{ccc} X & \xrightarrow{c} & \{R, B\} \\ f \downarrow & & \\ X & & \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{c} & \{R, B\} \\ f^{-1} \uparrow & \nearrow & \\ X & & \end{array}$$

composition $c \circ f^{-1}$ is a coloring of X

Define

$$f * c = c \circ f^{-1}$$

Thus

 $f * c$ is the coloring of X

that assigns each element $x \in X$ the
color $c(f^{-1}(x))$

Note f induces a perm of the set of all colorings of X :

$$\begin{aligned} \{ \text{coloring of } X \} &\rightarrow \{ \text{coloring of } X \} \\ c &\rightarrow f * c \end{aligned}$$

LEM For a coloring c of X
 $I * c = c$

I = identity elmt of G

$$\begin{aligned} \text{pf} \quad I * c &= c \circ I^{-1} \\ &= c \circ I \\ &= c \end{aligned}$$

LEM Given $f, g \in G$,

Given a coloring c of X ,

then

$$f * (g * c) = (f \circ g) * c$$

pf

$$\begin{aligned} f * (g * c) &= (g * c) \circ f^{-1} \\ &= (c \circ g^{-1}) \circ f^{-1} \\ &= c \circ (g^{-1} \circ f^{-1}) \\ &= c \circ (f \circ g)^{-1} \\ &= (f \circ g) * c \end{aligned}$$

□

LEM Given $f \in G$

Given colorings c_1 and c_2 of X

such that

$$c_2 = f * c_1$$

then

$$c_1 = f^{-1} * c_2$$

pf

$$\begin{aligned} f^{-1} * c_2 &= f^{-1} * (f * c_1) \\ &= (f^{-1} \circ f) * c_1 \\ &= I * c_1 \\ &= c_1 \end{aligned}$$

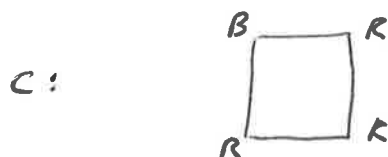
□

REX (i) For the coloring



$$f * C = C \quad \text{for all } f \in G$$

(ii) F_n



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Find $p * C$

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

1 2



R B

4 3

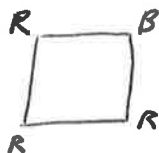
$p \downarrow \uparrow p^{-1}$



1 2

4 3

$p * C :$



"apply p to picture *

Find

 $\gamma * c$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

1 2



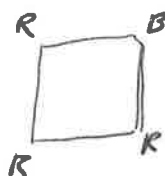
R B

4 3

 $\gamma \updownarrow$ 

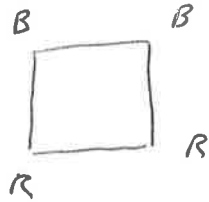
1 2

4 3

 $\gamma * c$ "apply γ to picture $*$ "

We record a principle

Given a coloring c of X , say



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Then for all $f \in G$,

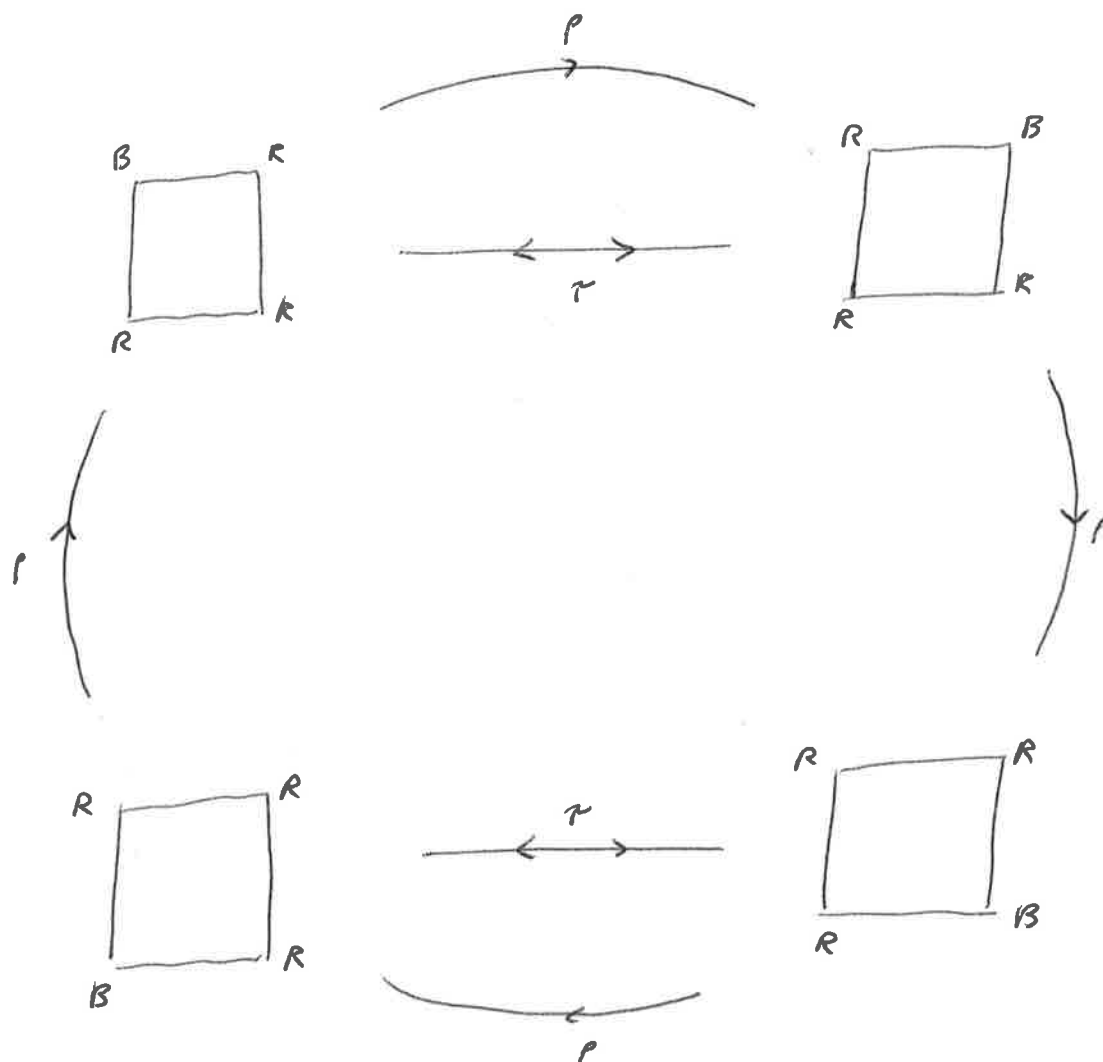
the coloring $f * c$ is obtained by applying f

to the picture *

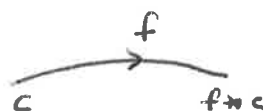
REX

Consider the set of colorings of X that have one B

G acts on this set as follows



key: For $f \in G$ and a coloring c of X ,



REX

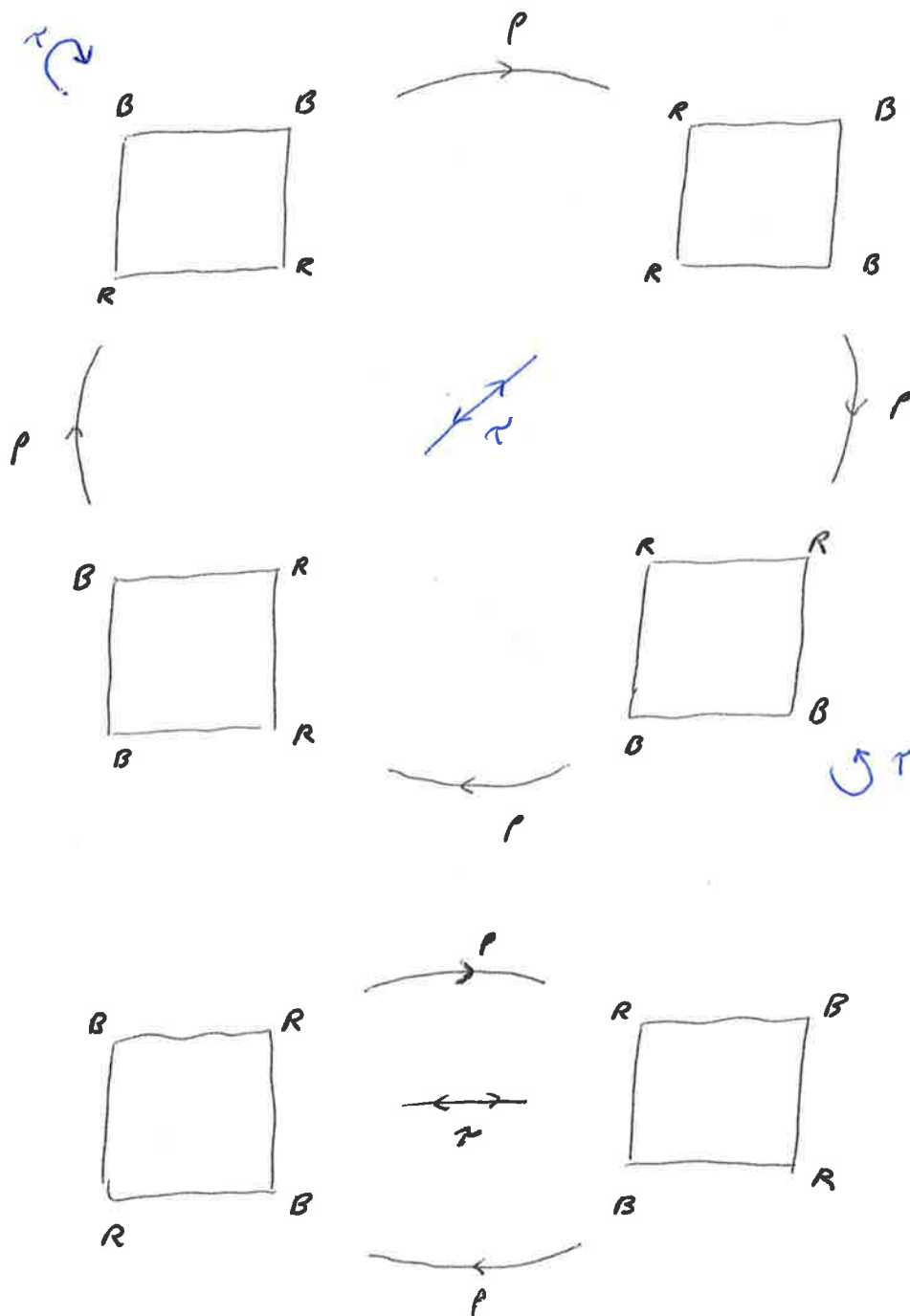
Consider the set of colorings of X

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that have two B's.

G acts on this set as follows:



Above diagram has 2 connected components.

Given colourings c_1, c_2 of X

call them G -equivalent and write $c_1 \overset{G}{\sim} c_2$

whenever they are in the same connected component of the diagram.

then

$$c_1 \overset{G}{\sim} c_2$$

means

there exists $f \in G$ such that

$$c_2 = f * c_1$$

The relation $\overset{G}{\sim}$ is an equivalence relation

and the equivalence classes are the connected components of the diagram.

Formal verification that $\sim^G$ is an equivalence

- For all colorings c of X show

$$c \sim^G c$$

pf

$$c = I * c$$

$I = \text{identity} \in G$

- For all colorings c_1, c_2 of X such that $c_1 \sim^G c_2$ show

$$c_2 \sim^G c_1$$

pf Since $c_1 \sim^G c_2 \exists f \in G$ s.t. $c_2 = f * c_1$

Now $c_1 = f^{-1} * c_2$ and $f^{-1} \in G$ so

$$c_2 \sim^G c_1$$

- For all colorings c_1, c_2, c_3 of X such that

$$c_1 \sim^G c_2 \quad \text{and} \quad c_2 \sim^G c_3$$

show

$$c_1 \sim^G c_3$$

pf

$$\exists f \in G \text{ s.t.}$$

$$c_2 = f * c_1$$

$$\exists g \in G \text{ s.t.}$$

$$c_3 = g * c_2$$

So

$$c_3 = g * c_2$$

$$= g * (f * c_1)$$

$$= \underbrace{(g \circ f)}_{\substack{\uparrow \\ G}} * c_1$$

So

$$c_1 \overset{G}{\sim} c_3$$

□

REX

Consider the set of the colourings

of X using colors R, B

The relation $\sim_G$ on this set has 6 equivalence classes, one for each row of table 1.

So up to G -equivalence, there are 6 colourings of X with R, B

— 0 —

