Math 846

Lecture 35

(We continue the proof of Thm 37)

Pick distinct $y, z \in X$

write $h = \partial(y, z)$.

We show

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} E\hat{w} = p^h_{ij} \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \left(E\hat{y} - E\hat{z} \right)$$

Since our base vertex x is arbitrary,

WLOG it suffices to show

coord x of LHS = coord x of RHS

To obtain this, compute (y,z)-entry in $(*)$: Lec 35

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$$(A_i A^* A_j)_{yz} = \sum_{w \in X} (A_i)_{yw} \, A^*_{ww} \, (A_j)_{wz}$$

$$= \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} A^*_{ww}$$

$$A^*_{ww} = |X| \, E_{xw} = |X| \left(x\text{-coord of } E\hat{w} \right)$$

$$= x\text{-coord of } \; |X| \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w}$$

The other terms are similar.

(ii) $\to$ (iii) clear

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(iii) → (i) We assume that E is nondeg,

so $\theta_i^* \neq \theta_0^* \quad (1 \le i \le D)$

Assume $D \ge 3$, since for $D=2$ Γ is Q-poly wrt E.

<u>Claim 1</u> Pick h $(1 \le h \le D)$ and $y, z \in X$ at $\partial(y,z) = h$. Then

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E\hat{w} - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E\hat{w} = r_{12}^h \left(E\hat{y} - E\hat{z}\right)$$

where

$$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}$$

<u>pf cl 1</u> By assumption $\exists\, \alpha \in \mathbb{F}$ st

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E\hat{w} - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E\hat{w} = \alpha \left(E\hat{y} - E\hat{z}\right)$$

In the above equation take the inner product of each term with $E\hat{y}$

Recall $\forall w \in X$

$$\langle E\hat{\omega}, E\hat{y}\rangle = |X|^{-1}\theta_j^* \quad j = \partial(w,y)$$

So

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} \langle E\hat{\omega}, E\hat{y}\rangle = |X|^{-1} p_{12}^h \theta_1^*$$

other terms similar. Get

$$|X|^{-1} p_{12}^h (\theta_1^* - \theta_2^*) = \alpha |X|^{-1} (\theta_0^* - \theta_h^*)$$

So $$\alpha = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}$$

claim proved ✓

<u>claim 2</u> We have

$$A A^* A_2 - A_2 A^* A = \sum_{h=1}^{D} r_{12}^h \left(A^* A_h - A_h A^*\right)$$

<u>pf cl 2</u> For $y, z \in X$ find (y,z)-entry of LHS$-$RHS

For $y = z$ one checks this entry is 0.

For $y \neq z$ this entry is

$$|X| \left\langle \sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E\hat{w} - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E\hat{w} - r_{12}^h (E\hat{y} - E\hat{z}) \, , \, E\hat{x} \right\rangle$$

$$h = \partial(y,z)$$

This expression is 0 by claim 1.

claim 2 proved ✓

Conceivably $\theta_1^* = \theta_2^*$

In this case $r_{12}^h = 0 \qquad 1 \le h \le D$

So by claim 2

$$A_2 A^* A = A A^* A_2$$

Use $$A_2 = \frac{A^2 - a_1 A - kI}{c_2}$$

To get

$$A^2 A^* A - A A^* A^2 = k(A^* A - A A^*)$$

We will return to this eqn shortly.

<u>claim 3</u> Suppose $\theta_1^* \ne \theta_2^*$, then $\exists\ \beta, \gamma, \delta \in \mathbb{F}$ st

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^*] \quad (\text{TD}1)$$

pf claim3 Recall

$$p^h_{12} = \begin{cases} 0 & \text{if } h>3 \\ \neq 0 & \text{if } h=3 \end{cases}$$

So

$$r^h_{12} = \begin{cases} 0 & \text{if } h>3 \\ \neq 0 & \text{if } h=3 \end{cases}$$

Also recall that for $0 \leq i \leq d$ $\exists\, f_i \in \mathbb{F}[\lambda]$

with $\deg f_i = i$

$A_i = f_i(A)$

In Claim 2, we write A_2, A_3 as a polynomial in A, and simplify to find

$$A^3A^* - A^*A^3 \in \operatorname{Span}\left(A^2A^*A - AA^*A^2,\ A^2A^* - A^*A^2,\ AA^* - A^*A \right)$$

So $\exists\, \beta, \gamma, \delta \in \mathbb{F}$ st

$$0 = A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

Rewrite this to get TD1. claim3 proved ✓

Recall the diagram Δ_E from the proof of the Pascasio thm.

Recall

- Δ_E is connected since we assume E is nondeg.
- node 0 is adjacent node 1 and no other node

Show Δ_E is a path

Δ_E :

To do this, we show that each node i of Δ_E is adjacent at most 2 nodes of Δ_E

Claim 4 Given distinct nodes $i, j \in \Delta_E$ that are adjacent,

if $\theta_i^* = \theta_2^*$ then $\theta_i \theta_j = -k$

if $\theta_i^* \neq \theta_2^*$ then $P(\theta_i, \theta_j) = 0$

where
$$P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

pf cl 4 First assume $\theta_1^* \neq \theta_2^*$. In TD1

multiply each term on left by E_i^* and on right by

E_j^*. Simplify to get

$$0 = \underbrace{E_i^* A^* E_j^*}_{\substack{\neq \\ 0 \\ \text{since } i,j \text{ adj} \\ \text{in } \Delta_E}} \underbrace{(\theta_i - \theta_j)}_{\substack{\neq \\ 0 \\ \text{since } i \neq j}} \underbrace{P(\theta_i, \theta_j)}_{\text{must be } 0}$$

So

$$P(\theta_i, \theta_j) = 0$$

Next assume $\theta_1^* = \theta_2^*$. From above cl 3,

$$A^2 A^* A - A A^* A^2 = k(A^* A - A A^*)$$

Mult each term on left by E_i^* and on right by E_j^*. Get

$$0 = \underbrace{E_i^* A^* E_j^*}_{\substack{\neq \\ 0}} \underbrace{(\theta_i - \theta_j)}_{\substack{\neq \\ 0}} \underbrace{(\theta_i \theta_j + k)}_{\text{must be } 0}$$

So

$$\theta_i \theta_j = -k$$

<u>claim 5</u> $\theta_1^* \neq \theta_2^*$

pf Suppose $\theta_1^* = \theta_2^*$. By claim 4 and since node 0 is adj node 1 in Δ_E,

$$\underset{\underset{k}{\parallel}}{\theta_0}\,\theta_1 = -k$$

so $\theta_1 = -1$

Δ_E is connected so node 1 is adj some node j $(j \neq 0)$

By claim 4,

$$\underset{\underset{-1}{\parallel}}{\theta_1}\,\theta_j = -k$$

so $\theta_j = k$

so $j = 0$, cont.

Claim 6 Each node i in Δ_E is adj at most 2 nodes in Δ_E

pf cl6 By claims 4, 5 for each node j of Δ_E that is adj node i, the eigenvalue θ_j is the root of

$$p(\theta_i, \mu) = \theta_i^2 - \beta\theta_i\mu + \mu^2 - \gamma(\theta_i + \mu) - \delta$$

This is a quadratic poly in μ so it has at most 2 dist roots. The claim follows. ✓

We have shown that Δ_E is a path, so Γ is Q-polynomial wrt E $\square$

Exercise $\underline{VI}$ Given a bipartite DRG $\Gamma = (X, \mathcal{R})$ diam $D \geq 2$. Assume the ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite Q-polynomial (so Γ is 2-homogeneous). Write $E = E_1$

Show the following hold for $0 \leq i, j \leq D$

- $$p^D_{ij} = \begin{cases} k_i = k_j & \text{if } i+j = D \\ 0 & \text{if } i+j \neq D \end{cases}$$

- For $y, z \in X$ at $\partial(y,z) = D$
$$\Gamma_i(y) \cap \Gamma_j(z) = \begin{cases} \Gamma_i(y) = \Gamma_j(z) & \text{if } i+j = D \\ \phi & \text{if } i+j \neq D \end{cases}$$

- For $y \in X$,
$$\sum_{w \in \Gamma_i(y)} E\hat{w} = \frac{k_i \theta_i^*}{\theta_0^*} E\hat{y}$$

- For $y, z \in X$ st $1 \leq \partial(y,z) \leq D-1$,
$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} = p^h_{ij} \frac{\theta_0^*\theta_i^* - \theta_h^*\theta_j^*}{\theta_0^{*2} - \theta_h^{*2}} E\hat{y} + p^h_{ij} \frac{\theta_0^*\theta_j^* - \theta_h^*\theta_i^*}{\theta_0^{*2} - \theta_h^{*2}} E\hat{z}$$

$h = \partial(y,z)$