

Combinatorics 475

Lecture 34

14.1 Cont.

 $X =$ nonempty finite set

Say

$$X = \{1, 2, \dots, n\}$$

Given perm

$$f: X \rightarrow X$$

$$f \neq I$$

Consider

$$I, f, f^2, f^3, \dots$$

X

Finitely many perms $X \rightarrow X$ Must be duplication among f^k :

$$f^r = f^s \quad r < s$$

so

$$f^{s-r} = I$$

so

$$\exists m \geq 1 \text{ such that}$$

$$f^m = I$$

Note

$$f^{-1} = f^{m-1}$$

Since

$$f \circ f^{m-1} = f^m = I$$

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DEF A permutation group on X is a set

G of perms $X \rightarrow X$ such that:

(1) For all $f, g \in G$

$$f \circ g \in G$$

" G is closed under composition"

(2) $I \in G$

LEM Given a perm $g \in G \subseteq S_X$.

Then for all $f \in G$

$$f^{-1} \in G$$

" G is closed under taking inverses"

pf

Case $f = I$: ok since $I^{-1} = I$

Case $f \neq I$: $\exists m \geq 1$ such that $f^m = I$

$$f^{-1} = f^{m-1}$$

$$= \underbrace{f \circ f \circ \dots \circ f}_{m-1}$$

$\in G$ since G closed under comp

□

Examples of Permutation Groups

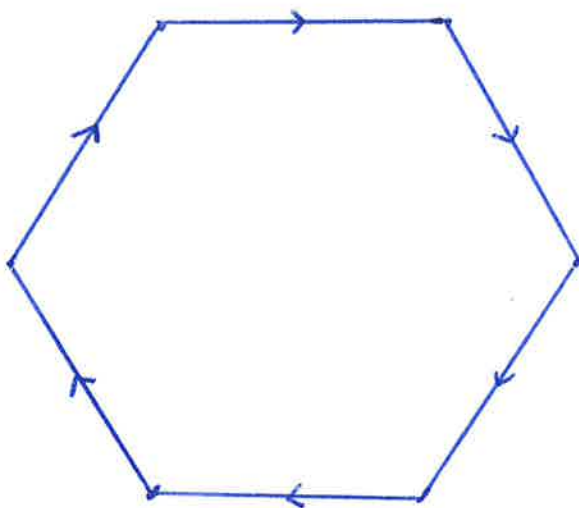
EX 1 Recall

 $S_n =$ set of all perms of $X = \{1, 2, \dots, n\}$ $S_n =$ perm group on X This group is called the symmetric group of order n

Ex 2 Consider an oriented regular n -gon P
in the plane

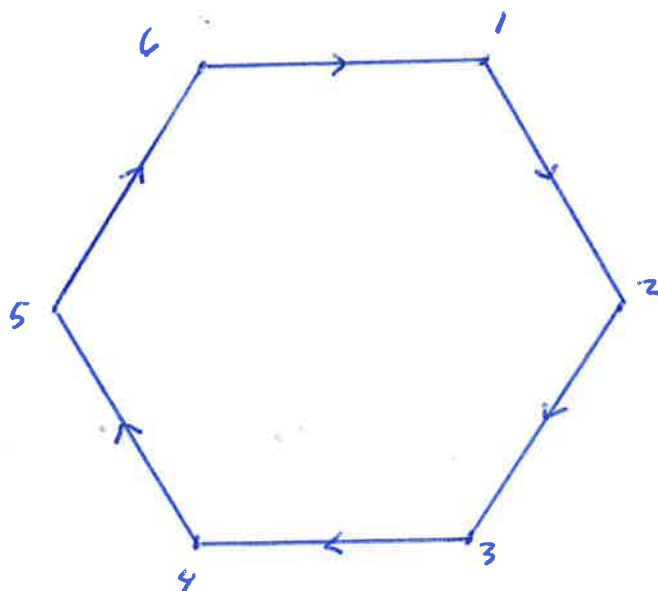
ex $n=6$

P :



View $X =$ set of corners (vertices) of P

P :



$$X = \{ 1, 2, 3, 4, 5, 6 \}$$

P has rotational symmetries:

If we rotate P clockwise by some multiple m of 60° ($0 \leq m \leq 5$)

then result coincides with P_0 .

Each rotation induces perm of X:

m	perm of X	name of perm
0	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$	I
1	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$	R
2	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$	$R^2 = R \circ R$
3	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$	R^3
4	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$	R^4
5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	R^5

Note $R^6 = I$

Define

$$G = \text{set of all rotational symmetries of } P$$

$$= \{ I, R, R^2, R^3, R^4, R^5 \}$$

$$\text{So } |G| = 6$$

Then G is a perm group on X

"cyclic group of order 6"

Inverses:

f	I	R	R^2	R^3	R^4	R^5
f^{-1}	I	R^5	R^4	R^3	R^2	R

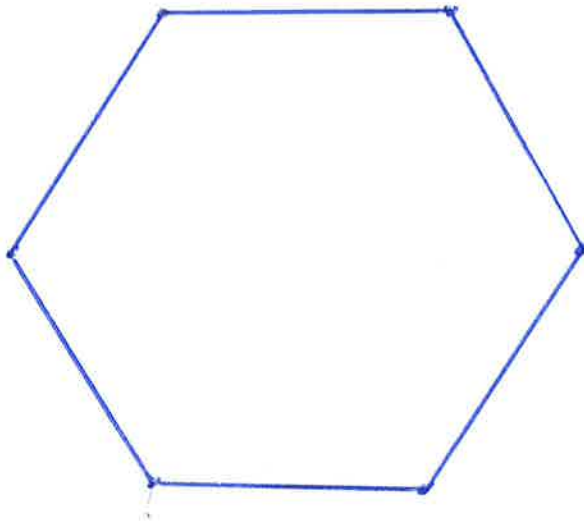
In general, for $n \geq 2$

the cyclic group of order n = group of rotational symmetries
of an oriented regular n -gon

Ex 3 Consider a non-oriented regular n -gon
in the plane

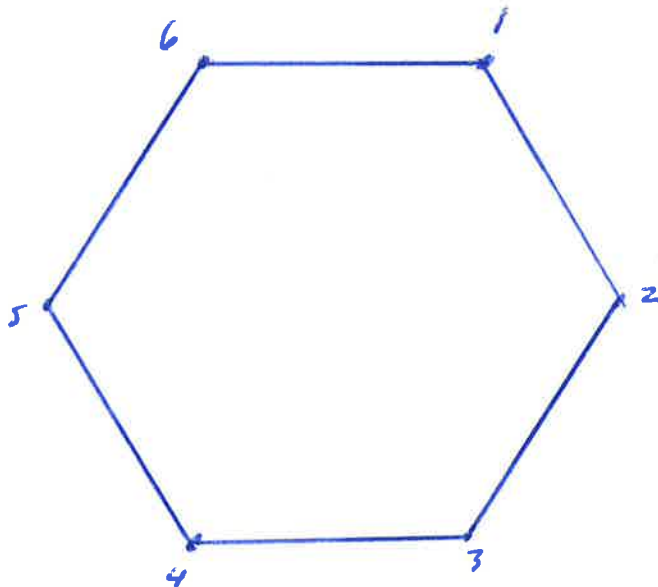
$$n=6$$

P :



View $X =$ sets of vertices of P

as before



$$X = \{1, 2, 3, 4, 5, 6\}$$

P has rotational and reflectonal symmetries

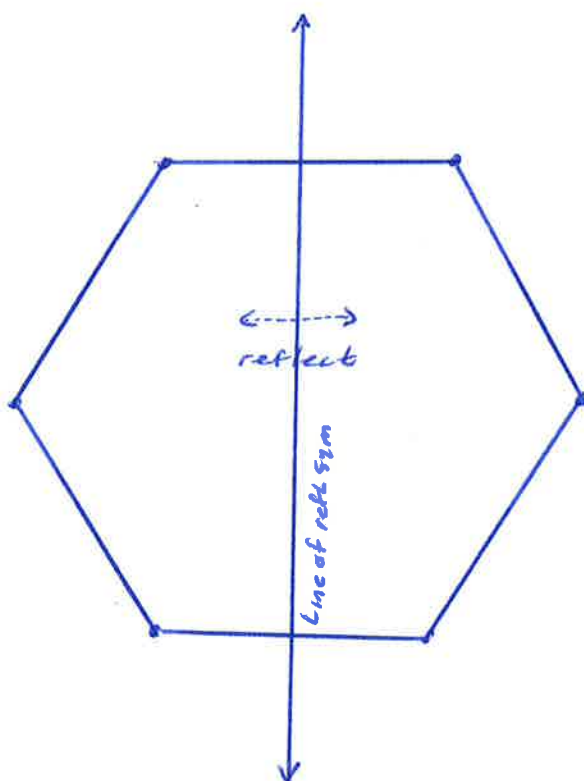
rotational symmetries : 6 of these just as in oriented case

reflectonal symmetries

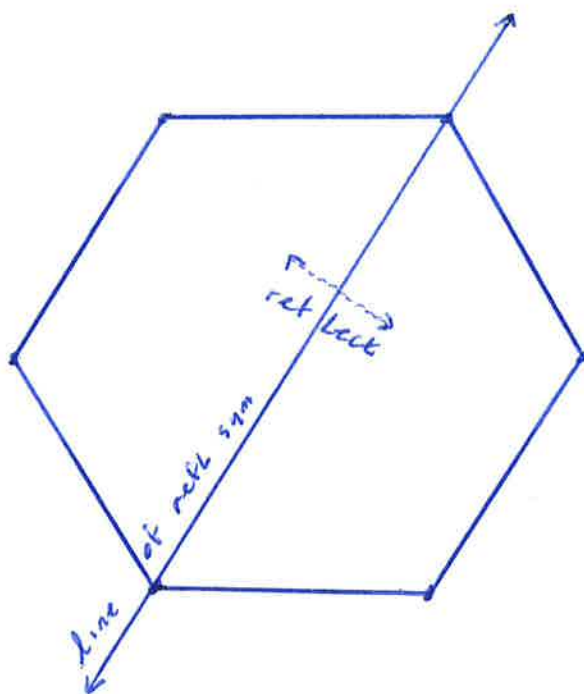
If we reflect P then a line of reflectonal symmetry,
the result coincides with P

ex

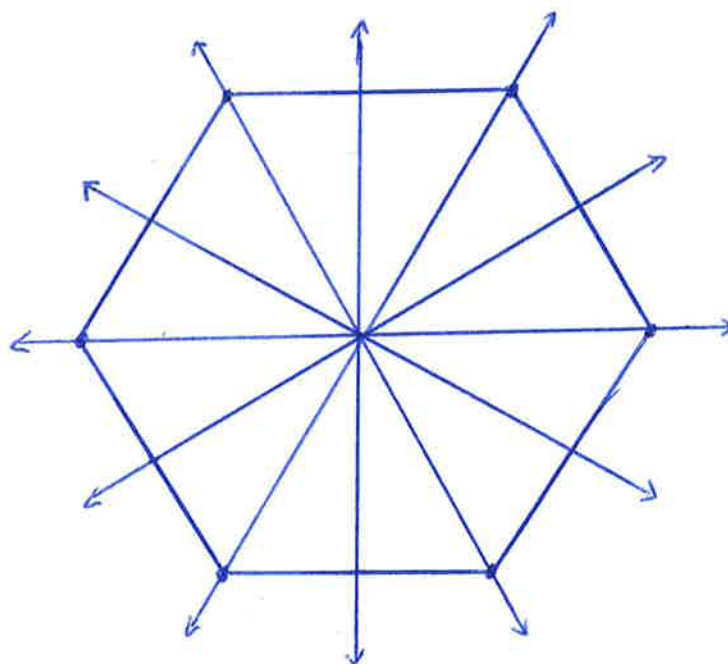
P :



P:



P has 6 lines of reflectional symmetry:



Each reflectional symmetry of P induces perm of X:

line of sym	perm of X	name
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$	τ
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$	$R \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$	$R^2 \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}$	$R^3 \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$	$R^4 \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$	$R^5 \circ \tau$

Recall $R^6 = I$

For any reflection

$$(\text{reflection})^2 = I$$

So

each reflection is its own inverse

obs

$$\tau^2 = I$$

$$\tau^{-1} = \tau$$

For $0 \leq i \leq 5$

$$(R^i \circ \tau)^2 = I$$

$$R^i \circ \tau = (R^i \circ \tau)^{-1}$$

$$= \tau^{-1} \circ R^{-i}$$

$$= \tau \circ R^{-i}$$

$$= \tau \circ R^{6-i}$$

Let $G =$ set of all symmetries of P , both rotational and reflectonal

$$= \{R^i\}_{i=0}^5 \cup \{R^i \circ \tau\}_{i=0}^5$$

$$|G| = 12$$

then G is a perm group on X

"dihedral group of order 12"

In general for $n \geq 3$

the dihedral group of order $2n$ = group of symmetries of the regular n -gon

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Any geometric figure of any dimension has a symmetry group

ex the 5 platonic solids in 3 dimensions

cube

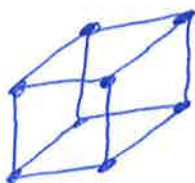
tetrahedron

octahedron

dodecahedron

icosahedron

cube



in 3 dimensions

X = set of vertices $|X| = 8$

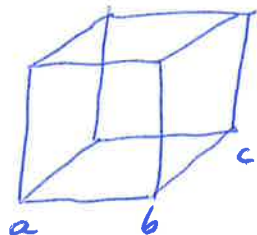
Each symmetry of the cube induces perm of X

let G = set of resulting perms of X

G = perm group on X

claim $|G| = 48$

pf consider 3 vertices



To construct $f \in G$ we define $f(a)$, $f(b)$, $f(c)$
in stages

stage	to do	#choices
1	Pick $f(a)$	8
2	Pick $f(b)$ from among the 3 vertices adjacent to a	3
3	Pick $f(c)$ from among 2 vertices adjacent to $f(b)$ other than $f(a)$	2

$$\begin{aligned} \#_{\text{poss}} &= 8 \times 3 \times 2 \\ &= 48 \end{aligned}$$

□

