Math 846
Lecture 34

Thm 34 (Brouwer Cohen Neumaier)

Assume Γ has classical parameters (D, b, α, σ).

Then

(i) $\theta = \frac{b_1}{b} - 1$ is an eigenvalue of Γ with $\theta \neq k$

(ii) Let

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

be the associated primitive idempotent. Then

$$\frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta}{k} - 1\right) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \qquad (0 \leq i \leq D)$$

(iii) Γ is Q-polynomial wrt E

pf (i), (ii) To show that θ is an eigenvalue of Γ, it suffices to check

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \qquad (0 \leq i \leq D)$$

where $\quad a_i = k - c_i - b_i \qquad 0 \leq i \leq D$

This check is routine.

Show $\theta \neq k$.

Suppose $\theta = k$

then

$$\frac{b_1}{b} - 1 = k$$

So $b > 0$

Using Def 33

$$b c_i - b_i - b(b c_{i-1} - b_{i-1}) = (k - \theta) b$$
$$= 0 \qquad (1 \le i \le D)$$

Hence

$$b c_i - b_i = b^i (b c_0 - b_0) \qquad (1 \le i \le D)$$
$$= -b^i k$$

Setting $i = D$ and $b_D = 0$

$$\underset{0}{\underset{\vee}{b c_D}} = \underset{0}{\underset{\vee}{-b^D k}}$$

cont. So $\theta \neq k$.

(iii) Check the conditions of Thm 31 are satisfied using

$$\beta = b + b^{-1}$$

$$\gamma^* = \theta_0^* \frac{\alpha(b^p - b) + \sigma(b-1) + b^2 - b}{k\, b}$$

$$\gamma = \frac{\alpha(b^p + 1) + \sigma(b-1) + 1 - b}{b}$$

$$w = \psi(\theta_1^* - \theta_0^*) - 2\gamma\theta_0^*$$

$$\eta^* = \gamma\, \theta_0^{*2} - \psi\, \theta_0^* (\theta_1^* - \theta_0^*)$$

where

$$\psi = 1 - \sigma - \frac{\alpha}{b}\begin{bmatrix} p+1 \\ 1 \end{bmatrix}$$

$\square$

LEM 35 Assume Γ has classical parameters (D, b, α, σ), and let $\{E_i\}_{i=0}^{D}$ denote the corresp Q-poly ordering of the primitive idempotents. Then

$$\theta_i^* = \frac{b_i}{b^i} - \left[\begin{matrix} i \\ 1 \end{matrix} \right] \qquad (0 \le i \le D)$$

pf We have

$$\theta_0 = k = b_0$$

$$\theta_1 = \frac{b_1}{b} - 1$$

then $\theta_2, \theta_3, \ldots, \theta_D$ are found using

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma \qquad (1 \le i \le D-1)$$

where

$$\beta = b + b^{-1}$$

$$\gamma = \frac{\alpha(b^D - 1) + \sigma(b - 1) + 1 - b}{b}$$

$\square$

Example 36 The following DRGs have classical parameters

(i) Hamming graph $H(D,N)$

$$b=1 \qquad \alpha=0 \qquad \sigma=N-1$$

(ii) Johnson graph $J(D,N)$ $(N\geq 2D)$

$$b=1 \qquad \alpha=1 \qquad \sigma=N-D$$

(iii) Grassmann graph $J_q(D,N)$ $(N\geq 2D)$

$$b=q \qquad \alpha=q \qquad \sigma=\frac{q^{N-D+1}-1}{q-1}-1$$

(iv) Bilinear forms graph $H_q(D,N)$ $(N\geq D)$

$$b=q \qquad \alpha=q-1 \qquad \sigma=q^N-1$$

pf For these examples the b_i, c_i are given in Ex 34 CH1
Compare these formula with Def 33. $\square$

Next goal: We give another characterization of the Q polynomial property, called the balanced set condition

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Given DRG $\Gamma = (X, R)$ with $D \geq 2$

We do not assume Γ is Q poly.

Given a nontrivial primitive idempotent E of Γ

write

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

Recall E is nondegenerate iff

$$\theta_i^* \neq \theta_0^* \qquad 1 \leq i \leq D$$

Thm 37 With the above notation TFAE:

(i) Γ is Q-polynomial wrt E

(ii) E is nondegenerate, and $\forall i, j$ $(0 \le i, j \le D)$ and $\forall$ distinct $y, z \in X$,

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} E\hat{w} = p^h_{ij} \frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h} \left(E\hat{y} - E\hat{z}\right)$$

$h = \partial(y, z)$

(iii) E is nondegenerate, and $\forall y, z \in X$,

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E\hat{w} - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E\hat{w} \in \operatorname{Span}\left(E\hat{y} - E\hat{z}\right)$$

pf Fix $x \in X$, write $T = T(x)$

Write $E = E_1$, $A^* = A_1^*$

(i) → (ii) E is nondeg since

$\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are mutually distinct.

Recall from Lem 5 in CH3,

$$\text{Span}\{ R A^* S - S A^* R \mid R, S \in M \}$$
$$= \{ Y A^* - A^* Y \mid Y \in M \}$$

(M = adjacency algebra)

Take $R = A_i$ and $S = A_j$. Get

$$A_i A^* A_j - A_j A^* A_i = \sum_{h=1}^{D} r_{ij}^h \left(A^* A_h - A_h A^* \right) \qquad (*)$$

for some $r_{ij}^h \in \mathbb{F}$

<u>claim</u> $$r_{ij}^h = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \qquad (1 \le h \le D)$$

<u>pf d</u> Fn $1 \le n \le D$ pick $z \in X$

at $\partial(x,z) = n$.

Compute (x,z)-entry in $(*)$

$$(A_i A^* A_j)_{xz} = \sum_{w \in X} (A_i)_{xw} (A^*)_{ww} (A_j)_{wz}$$

$$= \sum_{w \in \Gamma_i(x) \cap \Gamma_j(z)} \theta_i^*$$

$$= p^n_{ij} \, \theta_i^*$$

Similarly

$$(A_j A^* A_i)_{xz} = p^n_{ij} \, \theta_j^*$$

Similarly for $1 \le h \le D$

$$(A_h A^*)_{xz} = p^n_{h0} \, \theta_h^*$$

$$(A^* A_h)_{xz} = p^n_{0h} \, \theta_0^*$$

So by $(*)$

$$p^{\wedge}_{ij}(\theta_i^* - \theta_j^*) = \sum_{h=1}^{n} r^h_{ij}\,(\underset{\delta_{nh}}{\underset{\parallel}{p^{\wedge}_{0h}}}\,\theta_0^* - \underset{f_{hn}}{\underset{\parallel}{p^{\wedge}_{h0}}}\,\theta_h^*)$$

So

$$p^{\wedge}_{ij}(\theta_i^* - \theta_j^*) = r^{\wedge}_{ij}(\theta_0^* - \theta_n^*)$$

claim follows ✓