

Combinatorics 475

Lecture 33

Ch 14 Polya Counting

14.1 Permutations and symmetry groups

Let $X =$ nonempty finite set

Say $X = \{1, 2, \dots, n\}$

Consider perm σ of X :

$$a_1, a_2, \dots, a_n$$

View this as a bijection

$$\begin{array}{ccc} X & \longrightarrow & X \\ i & \longrightarrow & a_i \end{array}$$

To emphasize this view we often write

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

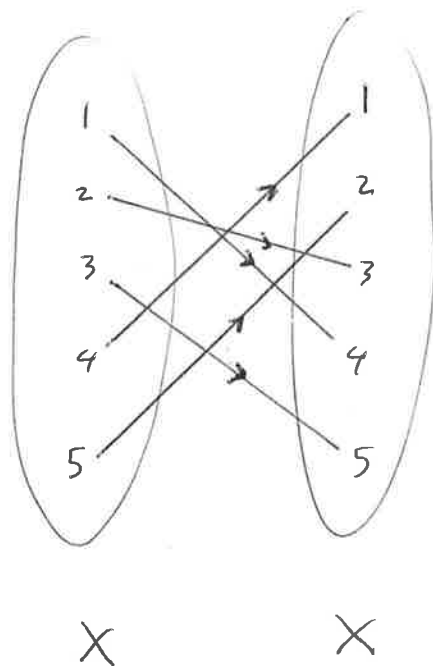
the bijection sends each number in top row to the number beneath it

Ex $n=5$

the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$$

satisfies



Def For $n \geq 1$

$S_n =$ set of all perms of $\{1, 2, \dots, n\}$

Composition of permutations

Given perms $f: X \rightarrow X$ $g: X \rightarrow X$

their composition $f \circ g: X \rightarrow X$ satisfies

$$(f \circ g)(x) = f(g(x)) \quad \forall x \in X$$

" First apply g and then apply f "



$$f \circ g: \quad x \rightarrow g(x) \rightarrow f(g(x))$$

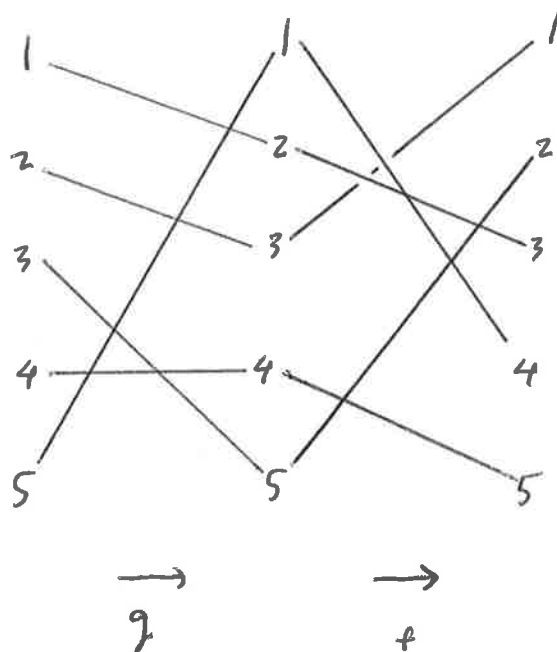
Ex for perms

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

Find $f \circ g$

Sol

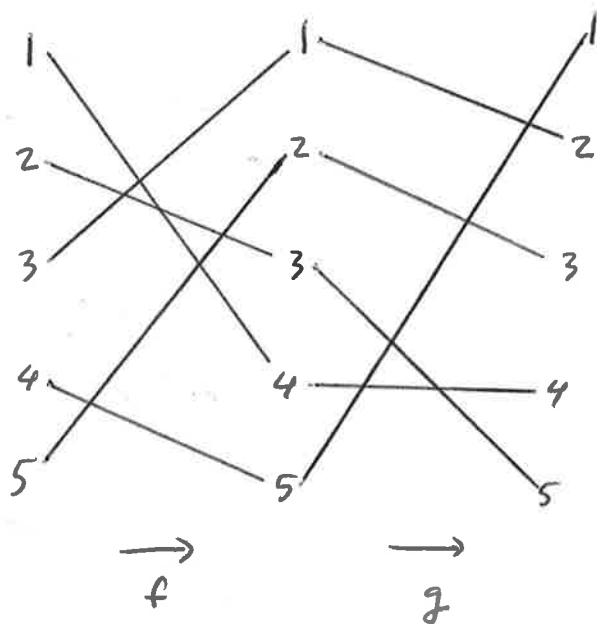


$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

$$\begin{aligned} (f \circ g)(1) &= f(g(1)) \\ &= f(2) \\ &= 3 \end{aligned}$$

etc.

Ex Referring to above f, g find $g \circ f$



$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

Note $g \circ f \neq f \circ g$ in general

We view composition $\circ$ as a binary operation

on S_n : given f and g in S_n their composition

$f \circ g$ is an element of S_n

LEM For $f, g, h \in S_n$

$$(f \circ g) \circ h = f \circ (g \circ h) \quad *$$

"composition is associative"

pf Each side of $*$ is a function $X \rightarrow X$

where $X = \{1, 2, \dots, n\}$

Show each function sends each $x \in X$ to the same thing

$$(f \circ g) \circ h : \quad \begin{array}{ccc} x & \xrightarrow{h} & h(x) \\ & \searrow & \downarrow \\ & & f \circ g \end{array} \longrightarrow (f \circ g)(h(x)) = f(g(h(x)))$$

"

$$f \circ (g \circ h) : \quad \begin{array}{ccc} x & \xrightarrow{g \circ h} & (g \circ h)(x) = g(h(x)) \\ & \searrow & \downarrow \\ & & f \end{array} \longrightarrow f(g(h(x)))$$

□

From now on we drop parenthesis and write

$$f \circ g \circ h$$

We abbreviate

$$f^2 = f \circ f, \quad f^3 = f \circ f \circ f, \quad f' = f$$

etc

DEF The identity permutation

$$I: X \rightarrow X$$

satisfies

$$I(x) = x$$

$$\forall x \in X$$

" I leaves everything alone "

So

$$I = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

LEM f a perm $f: X \rightarrow X$,

$$f \circ I = f$$

$$I \circ f = f$$

pf $\forall x \in X$

$$\begin{aligned} (f \circ I)(x) &= f(\underbrace{I(x)}_{=x}) \\ &= f(x) \end{aligned}$$

Also

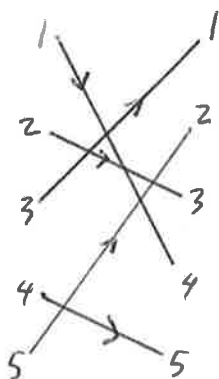
$$\begin{aligned} (I \circ f)(x) &= I(f(x)) \\ &= f(x) \end{aligned}$$

□

Inverses

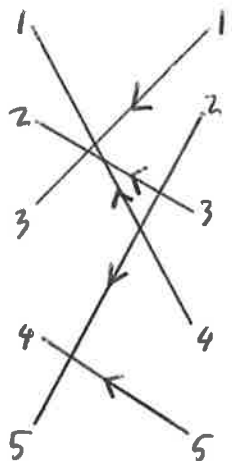
Given perm $f: X \rightarrow X$

say $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$



f

Change direction of arrows to get a new perm f^{-1}



f^{-1}

$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$$

Call f^{-1} the inverse of f

" f^{-1} undoes f "

Given a perm $f: X \rightarrow X$

then f^{-1} is described as follows:

View I

For all $x, y \in X$

$f(x) = y$ if and only if $f^{-1}(y) = x$

View II

For all $x \in X$

$$\text{ie } f^{-1} \circ f = I$$

$$f^{-1}(f(x)) = x$$

View III

For all $y \in X$

$$\text{ie } f \circ f^{-1} = I$$

$$f(f^{-1}(y)) = y$$

LEM Given perm $f: X \rightarrow X$

then for all perms $g: X \rightarrow X$

the following are equivalent:

$$(i) \quad g \circ f = I$$

$$(ii) \quad f \circ g = I$$

$$(iii) \quad g = f^{-1}$$

pf $(iii) \rightarrow (i)$ From View II alone

$(iii) \rightarrow (ii)$ From View III alone

$(i) \rightarrow (iii)$

$$g \circ f = I$$

$$(g \circ f) \circ f^{-1} = I \circ f^{-1}$$

"

"

$$g \circ (f \circ f^{-1})$$

$$f^{-1}$$

"

$$g \circ I$$

"

$$g$$

$(ii) \rightarrow (iii)$ Similar

□

Ex. Given perms

$$f: X \rightarrow X, \quad g: X \rightarrow X$$

find $(f \circ g)^{-1}$

Sol

$$f \circ g: \quad X \xrightarrow{g} X \xrightarrow{f} X$$

$f \circ g$ is undone by

$$(f \circ g)^{-1} \quad X \xleftarrow{g^{-1}} X \xleftarrow{f^{-1}} X$$

So

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

check

$$(g^{-1} \circ f^{-1}) \circ (f \circ g) \stackrel{?}{=} I$$

"

$$\underbrace{g^{-1} \circ \underbrace{f^{-1} \circ f}_{= I} \circ g}_{= I} = I$$

More generally, given perms

$$f_1: X \rightarrow X,$$

$$f_2: X \rightarrow X,$$

...

$$f_r: X \rightarrow X$$

$$(f_1 \circ f_2 \circ \dots \circ f_r)^{-1} = f_r^{-1} \circ \dots \circ f_2^{-1} \circ f_1^{-1}$$

In particular, for any perm

$$f: X \rightarrow X$$

and $r \geq 1$

$$(f^r)^{-1} = (f^{-1})^r$$

Call this commutative f^{-r}

Formally define

$$f^0 = I$$

By construction

$$f^r \circ f^s = f^{r+s}$$

for all integers r, s

$$(f^r)^s = f^{rs}$$

