Math 846

Lecture 33

Recall thm 31.

Pf We saw earlier that if Γ is Q-poly wrt E then E satisfies (i) – (iii).

Next we assume (i) – (iii) and show Γ is Q-poly rel E.

Fix $x \in X$ and write $T = T(x)$.

Write $E = E_1$, $A^* = A_1^*$.

For the time being, let $\{E_i\}_{i=2}^{D}$ denote any ordering of the remaining prim idempotents of Γ. Let

$$\theta_i = \text{eigvalue of } \Gamma \text{ for } E_i \qquad (0 \le i \le D)$$

Define a graph Δ_E with vertex set

$$\{0, 1, 2, \ldots, D\}$$

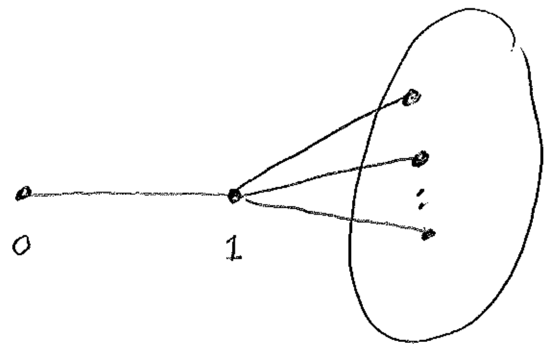
Vertices i, j are adjacent in Δ_E iff

$$i \ne j \quad \text{and} \quad q^1_{ij} \ne 0 \qquad (0 \le i, j \le D)$$

Recall $$q^1_{0j} = \delta_{1j} \qquad (0 \le j \le D)$$

So vertex 0 is adjacent to vertex 1 and no other vertices

Δ_E:



We will show that Δ_E is a path

<u>Claim 1</u> Δ_E is connected

<u>pf</u> Let $S \subseteq \{0, 1, \dots, D\}$ denote the connected component of Δ_E that contains 0, 1

Define $$U = \sum_{i \in S} E_i V$$ $V =$ st. module

Obs

$$E_1 V \subseteq U$$

Recall for $0 \le i \le D$

$$E_1 V \circ E_i V = \sum_{\substack{0 \le h \le D \\ q^1_{ih} \neq 0}} E_h V$$

So

$$E_1 V \circ U \subseteq U$$

E_1 is nondegenerate by axiom (i) so $E_1 V$ generates V in the function algebra $V, \circ$.

So $\quad U = V$

So $\quad S = \{0, 1, \ldots, D\}$

claim proved ✓

By constr (*) holds for $0 \le i \le D$.

So $\exists$ $\delta^* \in \mathbb{F}$ s.t.

$$\theta_{i-1}^{*2} - \beta \theta_{i-1}^* \theta_i^* + \theta_i^{*2} - \gamma^* (\theta_{i-1}^* + \theta_i^*) = \delta^*$$
$$(0 \le i \le D+1)$$

Using this one checks

$$(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = (2-\beta)\theta_i^{*2} - 2\gamma^* \theta_i^* - \delta^*$$
$$(0 \le i \le D)$$

<u>claim 2</u> On the primary T-module $e_0 V$

$$A^{*2}A - \beta A^* A A^* + A A^{*2} - \gamma^* (AA^* + A^* A) - \delta^* A$$
$$= \gamma A^{*2} + \omega A^* + \eta^* I$$

<u>pf cl</u> Let ℓ = LHS of above eqn

On the T-module $e_0 V$,

$$\ell = \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* \ell E_j^*$$

$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* A E_j^* \left(\theta_i^{*2} - \beta \theta_i^* \theta_j^* + \theta_j^{*2} - \gamma^* (\theta_i^* + \theta_j^*) - \delta^* \right)$$

$$= \sum_{i=0}^{D} E_i^* A E_i^* (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*)$$

$$= \sum_{i=0}^{D} E_i^* a_i (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*)$$

$$= \sum_{i=0}^{D} E_i^* (\gamma \theta_i^{*2} + \omega \theta_i^* + \eta^*)$$

$$= \gamma A^{*2} + \omega A^* + \eta^* I$$

claim proved

<u>claim 3</u> Given vertices $i, j \in \Delta_E$ at $\partial(i,j) = 2$. Assume there exists a unique vertex h in Δ_E that is adjacent to both i, j. Then

$$\gamma = \theta_i - \beta\theta_h + \theta_j$$

<u>pf cl</u> In the eqn of claim 2, mult each term on left by E_i and on right by E_j, and simplify. To help in this simplification note

$$E_i A^{*2} E_j = E_i A^* \left(\sum_{r=0}^{D} E_r \right) A^* E_j$$

$$= E_i A^* E_h A^* E_j$$

and

$$E_i A^* A A^* E_j = E_i A^* \left(\sum_{r=0}^{D} \theta_r E_r \right) A^* E_j$$

$$= E_i A^* E_h A^* E_j \quad \theta_h$$

By above comments,

$$0 \overset{\text{on } E_0V}{=} E_i A^* E_h A^* E_j (\theta_i - \beta\theta_h + \theta_j - \gamma)$$

But

$$0 \overset{\text{on } E_0V}{\neq} E_i A^* E_h A^* E_j$$

so

$$\gamma = \theta_i - \beta\theta_h + \theta_j.$$

claim proved ✓

We can now easily show Δ_E is a path.

Since Δ_E is connected, and since vertex 0 is adj only to vertex 1, it suffices to show that each vertex in Δ_E is adj at most 2 vertices in Δ_E. Suppose $\exists$ vertex i in Δ_E that is adj at least 3 vertices in Δ_E.

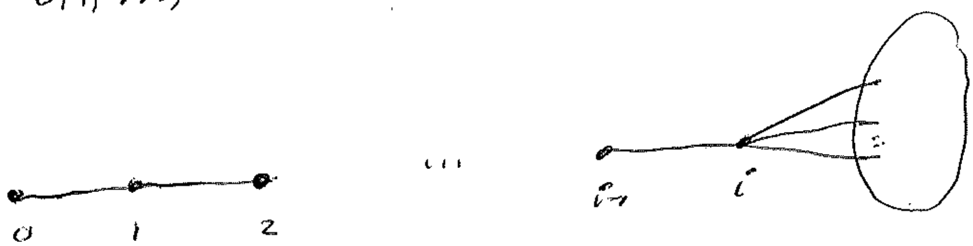
Of all such vertices pick i such that $\partial(0,i)$ is minimal.

WLOG the vertices of Δ_E are labelled

s.t. $\partial(0,i) = i$ and

$0, 1, \dots, i$ is a path in Δ_E :

Δ_E :



By constr $i \geq 1$

By assumption $\exists$ distinct vertices j, j' in $\Delta_E \setminus \{0, 1, \dots, i\}$

that are both adj to i

Obs $\partial(i-1, j) = 2$ and i is unique vertex in Δ_E adj

both $i-1, j$. So by Claim 3

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_j$$

Replacing j by j' in above argument,

$$\gamma = \theta_{i-1} - \beta\theta_i + \theta_{j'}$$

So $\theta_j = \theta_{j'}$ contr.

Therefore Δ_E is a path. Now Γ is Q-poly wrt E. $\square$

Note 32 Referring to Thm 31, assume Γ is Q-poly wrt E. Let $\{\theta_i\}_{i=0}^{D}$ denote the eigenvalue ordering for this Q-poly structure. These eigenvalues are found as follows.

- $\theta_0 = k$ (the valency)
- $\theta_1 =$ eigenvalue for E, found using

$$c_i \theta^*_{i-1} + a_i \theta^*_i + b_i \theta^*_{i+1} = \theta_1 \theta^*_i \qquad (0 \le i \le D)$$

- $\theta_2, \theta_3, \ldots, \theta_D$ are found recursively using

$$\theta_{i-1} - \beta \theta_i + \theta_{i+1} = \gamma \qquad (1 \le i \le D-1)$$

where β, γ are from Thm 31.

Ex Recall the Odd graph $\Gamma = O_{D+1}$ Lec 33 9.15

is a DRG with diam D and valency $k = D+1$

(i) $\theta = -D$ is an eigenvalue of Γ

(ii) let

$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

denote the prim idemp for θ. Then

$$\frac{\theta_i^*}{\theta_0^*} = \frac{(2D+1)^2(-1)^i - 2(2D+1)i(-1)^i - 1}{4D(D+1)} \qquad (0 \le i \le D)$$

(iii) Γ is Q-poly rel. E

(iv) let $\{E_i\}_{i=0}^{D}$ denote the corresp Q-poly ordering. Then

$$\theta_i = (-1)^i (D+1-i) \qquad (0 \le i \le D)$$

pf (i), (ii) Suf to check

$$c_i \theta^*_{i-1} + a_i \theta^*_i + b_i \theta^*_{i+1} = \theta \theta^*_i \qquad 0 \le i \le D$$

where $a_i = 0 \quad (0 \le i \le D-1), \qquad a_D = k - c_D$

check is routine.

(iii) Check the conditions of Thm 31 using

$\beta = -2$

$\gamma = 0$

$\omega = 0$

$\eta^* = 0$

$$\gamma^* = \frac{-\theta^*_0}{D(D+1)}$$

(iv) Use Note 32 $\square$

Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

We now define what it means for Γ to have

<u>classical parameters</u>.

Notation $\quad \forall\, b \in \mathbb{Z}$ write

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}_b = 1 + b + b^2 + \cdots + b^{i-1}$$

$$= \begin{cases} \frac{b^i - 1}{b-1} & \text{if } b \neq 1 \\ i & \text{if } b = 1 \end{cases}$$

Def 33 $\quad \Gamma$ has <u>classical parameters</u>

(D, b, α, σ) whenever the intersection numbers

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad 0 \leq i \leq D$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad 0 \leq i \leq D$$

Note in this case $\quad b \neq 0,\ b \neq -1$