

Combinatorics 475

Lecture 32

8.3 Partition numbers

Given integer $n \geq 1$

A partition of n is a representation of n
as a sum of pos integers

— sumation order unimportant. wlog list them
in noninc order

Partitions of $n = 5$:

	Notation
$1+1+1+1+1$	1^5
$2+1+1+1$	$2 1^3$
$2+2+1$	$2^2 1$
$3+1+1$	$3 1^2$
$3+2$	$3 2$
$4+1$	$4 1$
5	5

Def $\forall n \geq 1$

p_n = number of partitions of n

Take $p_0 = 1$

Given partition of n :

$$1^{a_1} 2^{a_2} \dots n^{a_n}$$

then

$$1 \cdot a_1 + 2a_2 + \dots + na_n = n$$

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View

$p_n =$ # of nonnegative integer solutions $a_1, a_2, \dots, a_n$ to *

We now consider the generating function

$$\sum_{n=0}^{\infty} p_n x^n$$

thm We have

$$\sum_{n=0}^{\infty} p_n x^n = \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \dots$$

$$= \prod_{k=1}^{\infty} (1-x^k)^{-1}$$

pf RHS =

$$(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\dots$$

$$= \left(\sum_{a_1=0}^{\infty} x^{a_1} \right) \left(\sum_{a_2=0}^{\infty} x^{2 \cdot a_2} \right) \left(\sum_{a_3=0}^{\infty} x^{3 \cdot a_3} \right) \dots$$

$$= \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \sum_{a_3=0}^{\infty} \dots x^{a_1 + 2a_2 + 3a_3 + \dots}$$

$$= \sum_{n=0}^{\infty} \left(\begin{array}{c} \# \text{ non neg integer solns to} \\ a_1 + 2a_2 + \dots + na_n = n \end{array} \right) x^n$$

$$= \sum_{n=0}^{\infty} p_n x^n$$

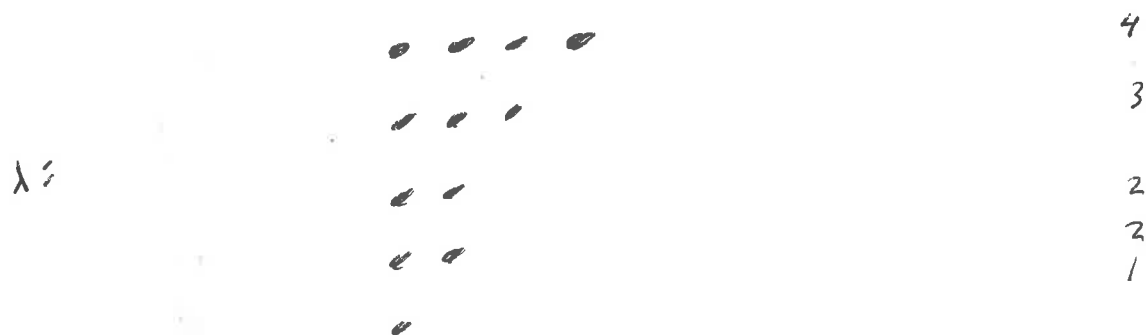
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The Ferrers diagram

Given a partition λ of n , say $n=12$

$$\lambda: 12 = 4 + 3 + 2 + 2 + 1$$

Rep by diagram:



The conjugate partition λ^* of n has the transposed diagram:



$$\lambda^* = 12 = 5 + 4 + 2 + 1$$

Def A partition λ of n is self-conjugate

whenever

$$\lambda^* = \lambda$$

ex the self-conjugate partitions of $n=8$ are

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Thm For $n \geq 1$ the following are equal:

(i) # of self-conjugate partitions of n

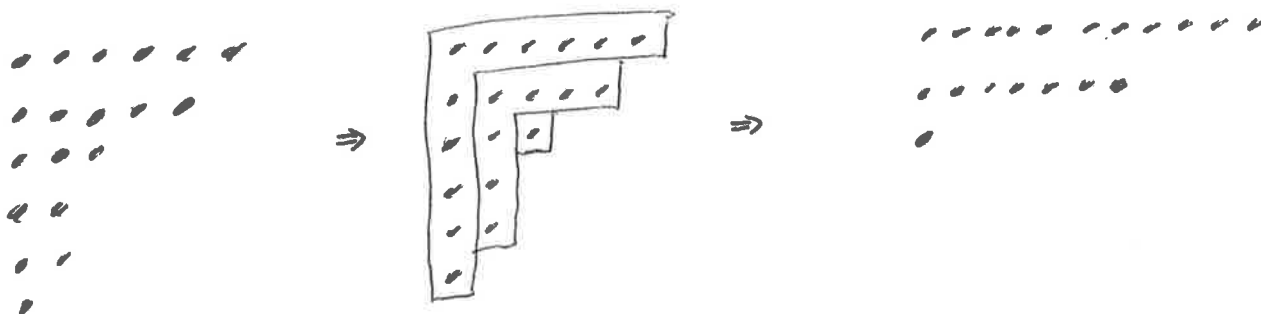
(ii) # of partitions of n into distinct odd parts

pf define

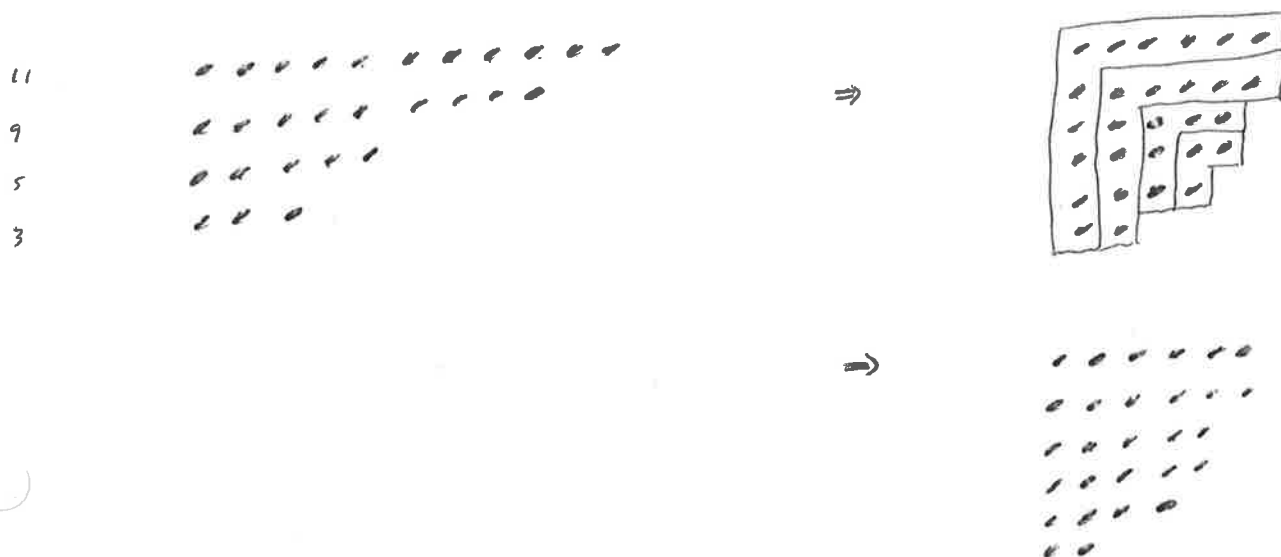
 $S_n =$ set of self-conjugate partitions of n $D_n =$ set of partitions of n into distinct odd parts

Display bijection

$$S_n \rightarrow D_n$$



inverse bijection:



Problem

Fr nzi Compare

partitions of n into odd parts

*

... distinct parts

**

Compare gen functions:

Gen function for * is

$$\frac{1}{1-x} \quad \frac{1}{1-x^3} \quad \frac{1}{1-x^5} \quad \dots$$

(1)

Gen function for ** is

$$(1+x)(1+x^2)(1+x^3) \dots$$

(2)

Obs (2) =

$$\frac{\cancel{1-x^2}}{1-x} \quad \frac{\cancel{1-x^4}}{\cancel{1-x^2}} \quad \frac{\cancel{1-x^6}}{1-x^3} \quad \frac{1-x^8}{\cancel{1-x^4}}$$

$$= \frac{1}{1-x} \quad \frac{1}{1-x^3} \quad \dots$$

$$= (1)$$

This shows:

Thm (Euler) For $n \geq 1$ the following are equal:

(i) # partitions of n into odd parts

(ii) # partitions of n into dist parts

ex $n=5$

partitions into odd parts

$$1 + 1 + 1 + 1 + 1$$

$$3 + 1 + 1$$

$$5$$

partitions into dist parts

$$3 + 2$$

$$4 + 1$$

$$5$$

Given $n \geq 1$

let $P_n =$ set of partitions of n

We give a partial order $\leq$ on P_n called
majorization (also called dominance order)

Given partitions λ, μ of n :

λ :

$$n = \lambda_1 + \lambda_2 + \lambda_3 + \dots$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

μ :

$$n = \mu_1 + \mu_2 + \mu_3 + \dots$$

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots$$

view as ∞ sequence with finitely many nonzero terms

Write

$$\lambda \geq \mu$$

whenever

$$\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i \quad \text{for } i = 1, 2, 3, \dots$$

Ex

$$n = 6$$

λ :

4
1
1

λ_i
4
1
1

$\lambda_1 + \dots + \lambda_i$
4
5
6

μ :

3
2
1

μ_i
3
2
1

$\mu_1 + \dots + \mu_i$
3
5
6

$$\lambda \geq \mu \quad \checkmark$$

E_x $n=6$ cont

λ $\begin{matrix} \bullet \bullet \bullet \\ \bullet \\ \bullet \\ \bullet \end{matrix}$

$\frac{\lambda_i}{3}$
1
1
1

$\frac{\lambda_1 + \dots + \lambda_i}{3}$
4
5
6

μ $\begin{matrix} \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \end{matrix}$

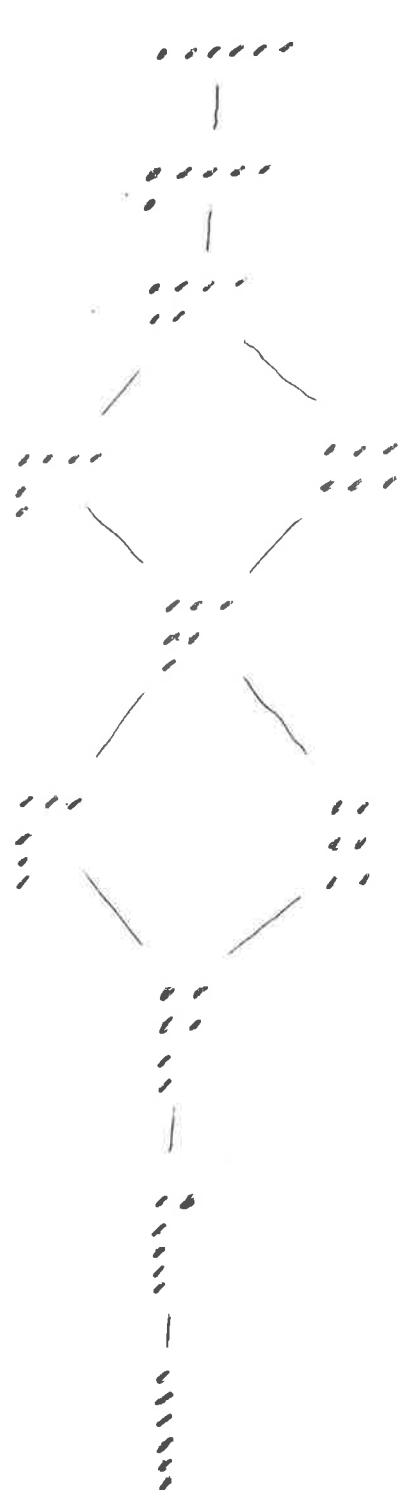
$\frac{\mu_i}{2}$
2
2
2
0

$\frac{\mu_1 + \dots + \mu_i}{2}$
4
6
6

$\lambda \not\leq \mu$ and $\mu \not\leq \lambda$

incomparable

$F_n = 6$ - We show the covering relation for majorization



max

min

 $p_6 = 11$

We have discussed the majorization

partial order on P_n

There is also a linear order on P_n called lexicographic order

LEX order on P_n :

Given partition λ of n

$$\lambda: n = \lambda_1 + \lambda_2 + \dots$$

$$\lambda_1 \geq \lambda_2 \geq \dots$$

View as 'word'

$$\lambda_1, \lambda_2, \dots$$

where 'letters'

$$\lambda_1, \lambda_2, \dots$$

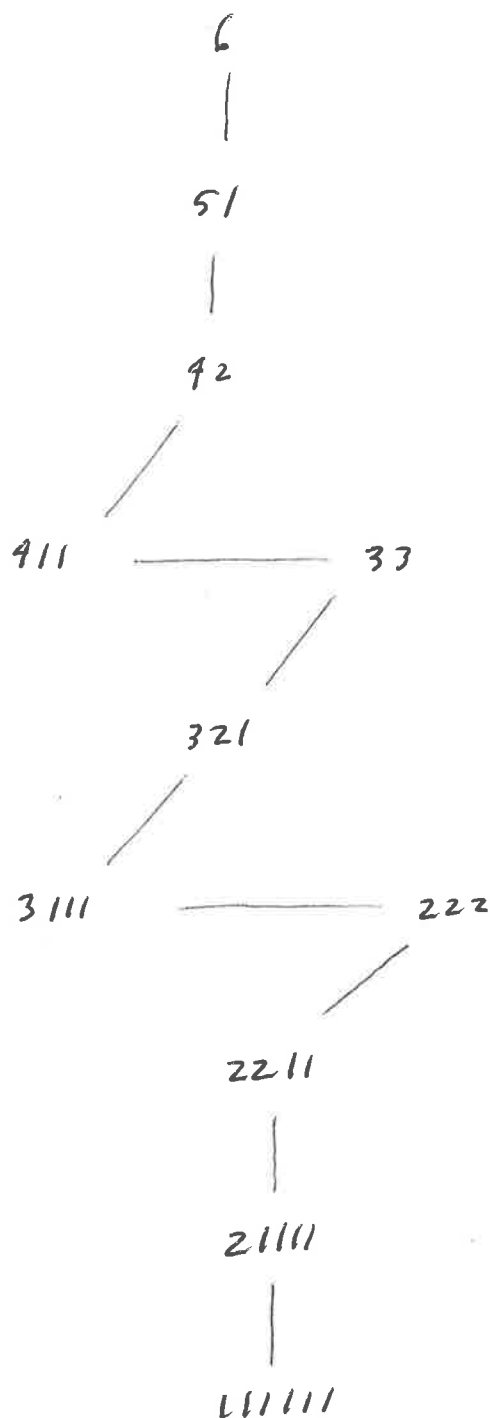
satisfy

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

LEX order is usual dictionary ordering of these words.

ex $n=6$ covering rel for LEX order:

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max

min

Obs: LEX order is a linear extension of majorization

Thm For $n \geq 1$ the LEX order on P_n

linearly extends the MAJ partial order on P_n

In other words, given partitions λ, μ of n such that

$$\lambda \leq_{\text{MAJ}} \mu$$

then

$$\lambda \leq_{\text{LEX}} \mu$$

p.f Assume $\lambda \neq \mu$ else triv.

Write

$\lambda:$

$$n = \lambda_1 + \lambda_2 + \dots$$

$$\lambda_1 \geq \lambda_2 \geq \dots$$

$\mu:$

$$n = \mu_1 + \mu_2 + \dots$$

$$\mu_1 \geq \mu_2 \geq \dots$$

Since $\lambda \neq \mu \exists i$

$$\lambda_i \neq \mu_i$$

Let

$$t = \min \{ i \mid \lambda_i \neq \mu_i \}$$

By constr

$$\lambda_i = \mu_i$$

$$1 \leq i \leq t-1$$

$$\lambda_1 + \lambda_2 + \dots + \lambda_t \leq \mu_1 + \mu_2 + \dots + \mu_t$$

so

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$$\lambda \leq \mu$$

But

$$\lambda \neq \mu$$

$$\lambda < \mu$$

now

$$\lambda < \mu$$

LEX

□

