Math 846

Lecture 32

Pf of thm 26

Define

$$\ell = A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*$$

By TD1,

$$[\ell, A] = 0$$

Aside on linear algebra: for any linear trans σ that is diagonalizable and has all eigenspaces of dimension 1, any linear trans that commutes with σ must be a polynomial in σ.

By this aside, $\exists\ f \in \mathbb{F}[\lambda]$ st

$$\ell \overset{\text{on } E_0V}{=\!=} f(A)$$

the minimal polynomial of A on E_0V has degree $D+1$.

So WLOG

$$\deg f \le D$$

let h = degree of f

<u>claim</u> $h \le 2$

<u>pf of cl</u> Suppose $h > 2$

Obs

$$\underbrace{E_h^* \, \ell \, E_0^*}_{\substack{\| \text{ by form of } \ell \\ 0}} \overset{\text{on } E_0V}{=\!=} \underbrace{E_h^* \, f(A) \, E_0^*}_{\substack{\| \\ \alpha \underbrace{E_h^* A^h E_0^*}_{\substack{\neq \text{ on } E_0V \\ 0}} \neq 0}}$$

α = leading coef of f

cont.

claim proved ✓

Write

$$\ell = \varepsilon A^2 + wA + zI, \qquad \varepsilon, w, z \in \mathbb{F}$$

show $\varepsilon = \gamma^*$.

First assume $D = 1$.

then I, A, A^2 lin dep, so ε can be chosen st

$$\varepsilon = \gamma^*$$

Next assume $D \geq 2$.

obs

$$\underbrace{E_2^* \ell E_0^*}_{\substack{\| \\ E_2^* A^2 E_0^* (\underbrace{\theta_0^* - \beta\theta_1^* + \theta_2^*}_{\substack{\| \\ \gamma^*}})}} \overset{\text{on } e_0 V}{=} \underbrace{E_2^* (\varepsilon A^2 + wA + zI) E_0^*}_{\substack{\| \\ \varepsilon E_2^* A^2 E_0^*}}$$

We have $E_2^* A^2 E_0^* \neq 0$ on $e_0 V$

so $\varepsilon = \gamma^*$

Interchanging the roles of A, A^* in the argument so far, $\exists\ w^*, z^* \in \mathbb{F}$ st

$$(A^*)^2 A - \beta A^* A A^* + A A^{*2} - \gamma^*(A^* A - A A^*) - \delta^* A$$
$$= \gamma A^{*2} + w^* A^* + z^* I \qquad (AW2')$$

Show $w = w^*$

Find the commutator of AW1 with A^*. Get

$$A^2 A^{*2} - \beta A A^* A A^* + \beta A^* A A^* A - A^{*2} A^2$$
$$- \gamma (A A^{*2} - A^{*2} A)$$
$$\overset{mod V}{=} \gamma^* (A^2 A^* - A^* A^2) + w (A A^* - A^* A)$$

Next find the commutator of AW2' with A. Get

$$A^2 A^{*2} - \beta A A^* A^* A + \beta A A^* A A^* - A^2 A^{*2}$$
$$- \gamma^* (A^* A^2 - A^2 A^*)$$
$$\overset{mod V}{=} \gamma (A^{*2} A - A A^{*2}) + w^* (A^* A - A A^*)$$

Adding the above eqns, get

$$0 \overset{\text{on } e_0 V}{=} (w - w^*)(AA^* - A^*A)$$

We have

$$AA^* \overset{\text{on } e_0 V}{\neq} A^*A$$

otherwise $e_0 V$ would not be irreducible as a T-module. So

$$w = w^* \qquad \square$$

Next we consider how to find

w, z, z^*.

<u>Notation</u> Recall

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma \qquad (1 \le i \le p-1)$$

$$\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = \gamma^* \qquad (1 \le i \le p-1)$$

Define $\theta_{-1}, \theta_{p+1}, \theta_{-1}^*, \theta_{p+1}^*$

so that above eqns hold at $i = 0$ and $i = p$

So $\theta_{-1} - \beta\theta_0 + \theta_1 = \gamma$.

etc.

Lem 28 For $0 \le i \le D$,

(i) $p(\theta_i, \theta_i) = (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1})$

(ii) $p^*(\theta_i^*, \theta_i^*) = (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*)$

pf (i) First assume $1 \le i \le D$. Elim θ_{i+1} in RHS using

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma$$

and evaluate the result using

$$p(\theta_i, \theta_i) = (2-\beta)\theta_i^2 - 2\gamma\theta_i - \delta,$$

$$p(\theta_{i-1}, \theta_i) = 0$$

Next assume $0 \le i \le D-1$. Elim θ_{i-1} in RHS using

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma$$

and evaluate the result using

$$p(\theta_i, \theta_i) = (2-\beta)\theta_i^2 - 2\gamma\theta_i - \delta$$

$$p(\theta_i, \theta_{i+1}) = 0$$

$\square$

(ii) Similar.

Thm 29 With above notation,

(i) $$w = a_i^*(\theta_i - \theta_{i+1}) + a_{i-1}^*(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_{i-1} + \theta_i) \qquad (1 \le i \le D)$$

(ii) $$w = a_i(\theta_i^* - \theta_{i+1}^*) + a_{i-1}(\theta_{i-1}^* - \theta_{i-2}^*) - \gamma(\theta_{i-1}^* + \theta_i^*) \qquad (1 \le i \le D)$$

(iii) $$\eta = a_i^*(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - w\,\theta_i - \gamma^*\theta_i^2 \qquad (0 \le i \le D)$$

(iv) $$\eta^* = a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - w\,\theta_i^* - \gamma\,\theta_i^{*2} \qquad (0 \le i \le D)$$

pf We start with (iii)

(iii) By Thm 26 and Lem 27

(iv) Similar to (iii)

(i) Subtract (iii) (at i) from (iii) (at $i-1$)

(ii) Similar to (i)

□

Until further notice

we are given a DRG $\Gamma = (X, R)$ with

$D \geq 2$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

We do not assume that Γ is Q polynomial

Next goal: Find an easy way to determine if Γ is Q polynomial or not.

Def 30 With above notation, let E denote a nontrivial primitive idempotent of Γ.

We say that Γ is Q polynomial with respect to E whenever there exists an ordering $\{E_i\}_{i=1}^{D}$ of the nontrivial primitive idempotents of Γ s.t $E = E_1$.

Thm 31 (Arlene Pascasio)

Given a DRG $\Gamma = (X, R)$ with $D \geq 2$.

Let
$$E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$$

denote a nontrivial primitive idempotent of Γ

Then Γ is Q-polynomial with respect to E iff

(i) $\theta_i^* \neq \theta_0^* \quad (1 \leq i \leq D)$

(ii) $\exists\ \beta, \gamma^* \in \mathbb{F}$ st
$$\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* = \gamma^* \quad (1 \leq i \leq D-1) \qquad (*)$$

(iii) $\exists\ \gamma, \omega, \eta^* \in \mathbb{F}$ st

$$a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = \gamma \theta_i^{*2} + \omega \theta_i^* + \eta^* \quad (0 \leq i \leq D)$$

where θ_{-1}^*, θ_{D+1}^* are defined st $(*)$ holds at $i = 0$ and $i = D$.