

Combinatorics 475

Lecture 31

DEF For $n \geq 0$ define

$$B_n = \sum_{k=0}^n S(n, k)$$

"nth Bell number"

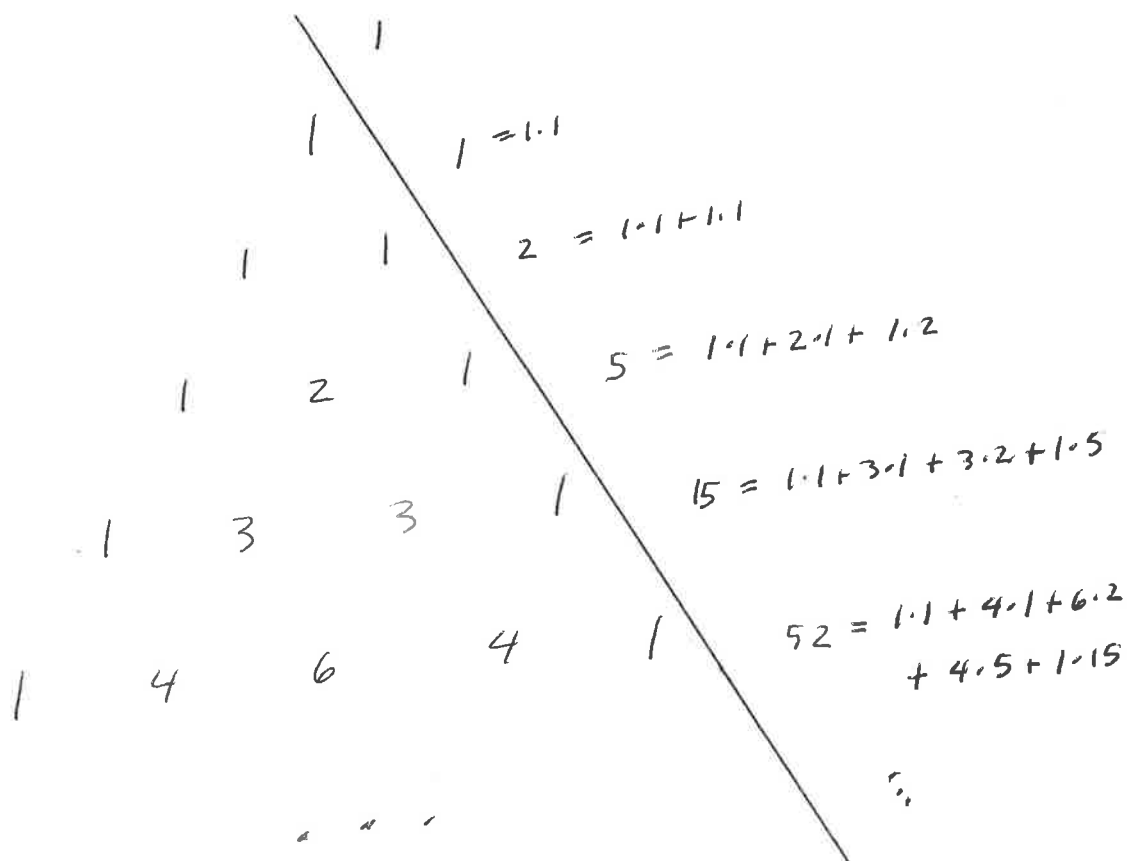
n	$S(n, k)$					B_n
0			1			1
1		0		1		1
2		0	1		1	2
3		0	1	3		5
4		0	1	7	6	15
5	0	1	15	25	10	52
$\vdots$			$\dots$			$\vdots$

By the prev Cor

$B_n = \# \text{ ways to place } n \text{ distinguishable objects into } n \text{ empty indistinguishable boxes}$

$$B_n \searrow$$

weights

$$(\hat{k})$$


Thm For $n \geq 1$

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

Pf

let $S =$ set of all partitions of $\{1, 2, \dots, n\}$ into
nonempty indistinguishable boxes

$$|S| = B_n$$

Given a partition p in S

let
index of $p = \left| \left\{ i \mid 1 \leq i \leq n-1, i \text{ not in same box as } n \right\} \right|$
 $0 \leq \text{index} \leq n-1$

For $0 \leq k \leq n-1$ define

$S_k =$ set of partitions in S with index k

$$S = \bigcup_{k=0}^{n-1} S_k \quad (\text{disjoint union})$$

so

$$|S| = \sum_{k=0}^{n-1} |S_k|$$

For $0 \leq k \leq n-1$ find $|S_k|$

To construct an element of S_k we proceed in stages

stage	to do	# choices
1	pick a k -subset X of $\{1, 2, \dots, n\}$	$\binom{n-1}{k}$
2	partition X into nonempty indist boxes	B_k
3	add another box containing n and $\{1, 2, \dots, n\} \setminus X$	1

$$S_0 \quad |S_k| = \binom{n-1}{k} B_k$$

$$S_0 \quad B_n = |S| = \sum_{k=0}^{n-1} |S_k|$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

□

Stirling numbers of 1st kind

Write $\{[x]_n\}_{n=0}^{\infty}$ in terms of $\{x^n\}_{n=0}^{\infty}$

Ex $[x]_3$ in terms of $\{x^n\}_{n=0}^{\infty}$

$$[x]_3 = x(x-1)(x-2)$$

$$= x^3 - 3x^2 + 2x$$

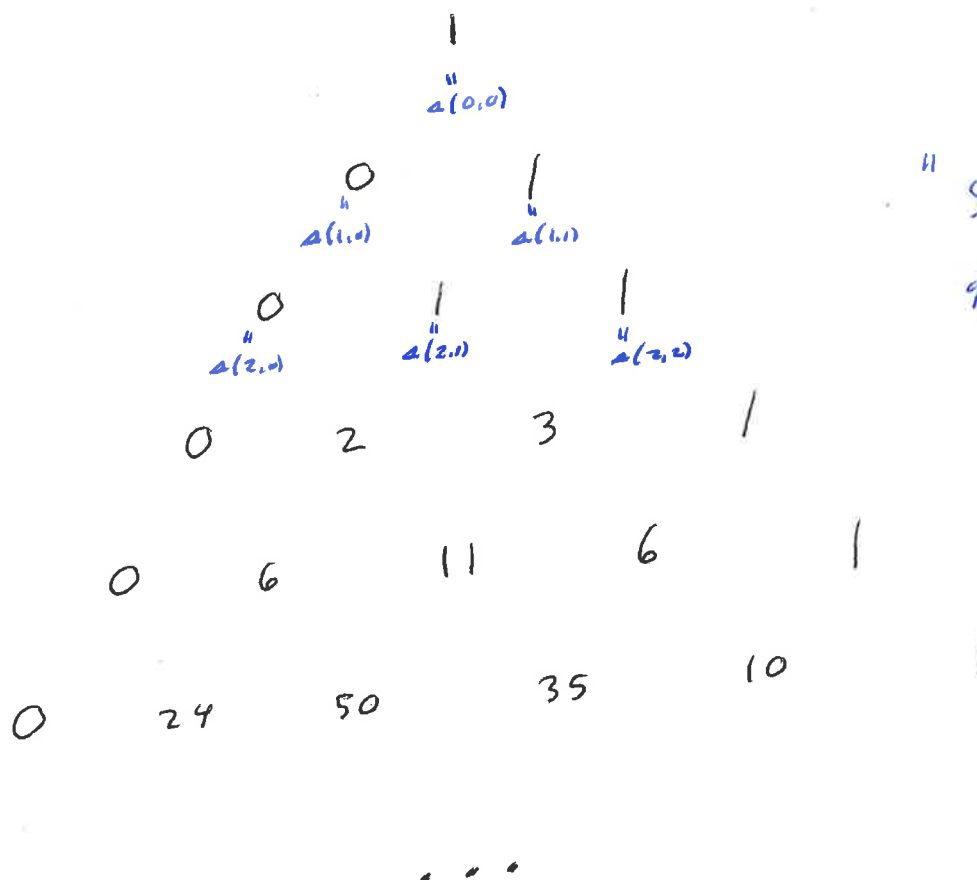
$$= 0 \cdot 1 + 2 \cdot x - 3 \cdot x^2 + 1 \cdot x^3$$

More generally we have

	x^0	x^1	x^2	x^3	x^4	x^5
$[x]_0$	1					
$[x]_1$	0	1				
$[x]_2$	0	-1	1			
$[x]_3$	0	2	-3	1		
$[x]_4$	0	-6	11	-6	1	
$[x]_5$	0	24	-50	35	-10	1
$\vdots$			...			

the signs alternate — we ignore them

View as analogy of Pascal triangle



" Stirling numbers
of 1st kind "

Stirling numbers	
$\Delta(n, k)$	1st kind
$S(n, k)$	2nd kind

By construction

$$[X]_n = \sum_{k=0}^n (-1)^{n-k} \Delta(n, k) x^k$$

$n = 0, 1, 2, \dots$

Obs

$$\Delta(n, 0) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

$$\Delta(n, 1) = (n-1)!$$

$n = 1, 2, \dots$

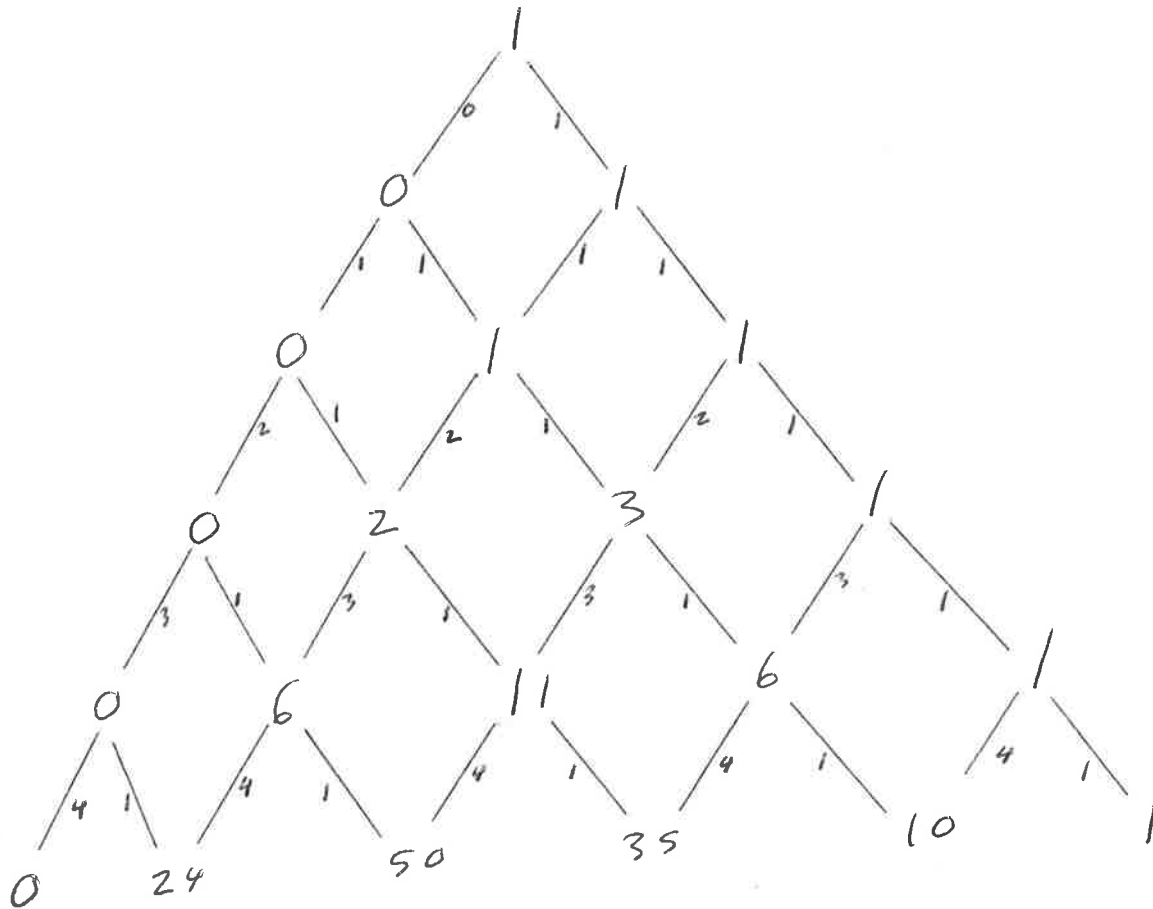
$$\Delta(n, n) = 1$$

$n = 0, 1, 2, \dots$

Next goal: show $a(n,k)$ satisfy a recurrence
similar to Pascal's formula

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...

For notational convenience define

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$$\Delta(n, k) = 0 \quad \text{if } k < 0 \text{ or } k > n$$

Thm for $n \geq 1$ and $0 \leq k \leq n$

$$\Delta(n, k) = \Delta(n-1, k-1) + (n-1)\Delta(n-1, k)$$

pf [Assume $n \geq 1$ else trivial]

Recall

$$[X]_n = \sum_{k=0}^n (-1)^{n-k} \Delta(n, k) x^k$$

☆

So

$$[X]_{n-1} = \sum_{k=0}^{n-1} (-1)^{n-1-k} \Delta(n-1, k) x^k$$

Obs

$$\begin{aligned} [X]_n &= x(x-1)(x-2) \cdots (x-n+2)(x-n+1) \\ &= [X]_{n-1} (x-n+1) \end{aligned}$$

$$= \sum_{k=0}^{n-1} (-1)^{n-1-k} \Delta(n-1, k) x^k (x-n+1)$$

$$= \sum_{k=0}^{n-1} (-1)^{n-k} a(n, k) x^{k+1}$$

$\leftarrow k+1$

$$- (n-1) \sum_{k=0}^{n-1} (-1)^{n-k} a(n, k) x^k$$

$$= \sum_{k=1}^n (-1)^{n-k} a(n, k) x^k$$

$\downarrow$
0

$$- (n-1) \sum_{k=0}^n (-1)^{n-k} a(n, k) x^k$$

$$= \sum_{k=0}^n (-1)^{n-k} \left(a(n, k+1) + (n-1) a(n, k) \right) x^k$$

Compare this to $\star$ to get

$$a(n, k) = a(n, k+1) + (n-1) a(n, k)$$

ask 2n

□

Next goal: Show that for $0 \leq k \leq n$

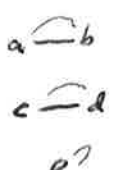
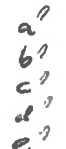
$a(n, k) =$ # arrangements of n distinguishable objects
into k nonempty circular permutations.

ex $n=5$

$a(n,k) = \#$ arrangements of n dist objects
into k nonempty circular permutations

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k	desc	$a(5,k)$
0	ϕ	0
1		$4! = 24$
2	 	$5 \times 3!$ $+ \binom{5}{2} 2!$ $= 50$
3	 	$\binom{5}{2} \times 2!$ $+ 5 \times 3$ $= 35$
4		$\binom{5}{2} = 10$
5		1

thm for $0 \leq k \leq n$

$a(n, k) = \#$ arrangements of n distinguishable objects
into k nonempty circular permutations

pf Ind on n

$$n = 0 \checkmark$$

$$n \geq 1 :$$

Call the objects $1, 2, \dots, n$

Define

$S =$ set of all arrangements of $1, 2, \dots, n$ into
 k nonempty circular permutations

Show $|S| = a(n, k)$

Define

$S' =$ set of elements in S for which the circular
perm containing n has length 1

$S'' =$ complement of S' in S

$$S = S' \cup S''$$

(disj union)

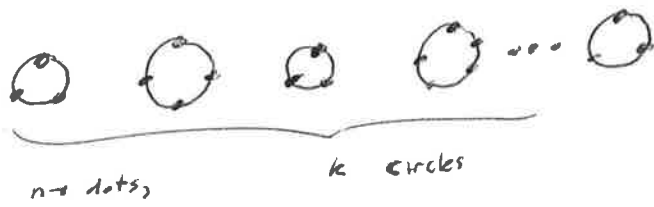
$$|S| = |S'| + |S''|$$

$|S'| = \# \text{ of arrangements of } 1, 2, \dots, n \text{ into } k \text{ nonempty circular perms}$

$$= a(n, k)$$

Find $|S''|$

We construct an element of S'' in stages

stage	to do	# choices
1	pick an arrangement of $1, 2, \dots, n$ into k nonempty circular perms 	$a(n, k)$
2	insert " n "	$n-1$

$$|S''| = (n-1)a(n, k)$$

$$\begin{aligned}
 \text{so } |S| &= |S'| + |S''| \\
 &= a(n, k) + (n-1)a(n, k) \\
 &= a(n, k)
 \end{aligned}$$



