Math 846

Lecture 31

We continue to discuss a DRG $\Gamma = (X, R)$ with $D \geq 1$.

Fix $x \in X$ and write $T = T(x)$

Referring to Lem 21, as special cases we mention

$$\hat{x} = \mathbb{1}_0$$
$$= |X|^{-1} \sum_{i=0}^{D} \mathbb{1}_i^*$$

and

$$\mathbb{1} = \mathbb{1}_0^*$$
$$= \sum_{i=0}^{D} \mathbb{1}_i$$

We emphasize the role of the polynomials

LEM 22

(i) We have

$$\mathbb{1}_i = v_i(A)\mathbb{1}_0 \qquad (0 \le i \le D)$$

(ii) Assume $\{E_i\}_{i=0}^{D}$ is Q-polynomial. Then

$$\mathbb{1}_i^* = v_i^*(A^*)\mathbb{1}_0^* \qquad (0 \le i \le D)$$

pf (i) Use

$$\mathbb{1}_i = A_i \hat{x}$$

$$\hat{x} = \mathbb{1}_0$$

$$A_i = v_i(A)$$

(ii) Use

$$\mathbb{1}_i^* = A_i^* \mathbb{1}$$

$$\mathbb{1} = \mathbb{1}_0^*$$

$$A_i^* = v_i^*(A^*)$$

$\square$

We mention a special case of Lem 19, 20.

LEM 23

(i) With respect to the bases $\{\mathbb{1}_i\}_{i=0}^{D}$ and $\{\mathbb{1}_i^*\}_{i=0}^{D}$ the matrices representing A are

$$\begin{pmatrix} a_0 & b_0 & & & \bigcirc \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{D-1} \\ \bigcirc & & & c_D & a_D \end{pmatrix}, \qquad \begin{pmatrix} \theta_0 & & & \bigcirc \\ & \theta_1 & & \\ & & \ddots & \\ \bigcirc & & & \theta_D \end{pmatrix}$$

respectively.

(ii) Assume the ordering $\{E_i\}_{i=0}^{D}$ is Q-poly. Then with respect to the bases $\{\mathbb{1}_i\}_{i=0}^{D}$ and $\{\mathbb{1}_i^*\}_{i=0}^{D}$ the matrices representing A^* are

$$\begin{pmatrix} \theta_0^* & & & \bigcirc \\ & \theta_1^* & & \\ & & \ddots & \\ \bigcirc & & & \theta_D^* \end{pmatrix}, \qquad \begin{pmatrix} a_0^* & b_0^* & & & \bigcirc \\ c_1^* & a_1^* & b_1^* & & \\ & c_2^* & \ddots & \ddots & \\ & & \ddots & \ddots & b_{D-1}^* \\ \bigcirc & & & c_D^* & a_D^* \end{pmatrix}$$

respectively.

pf clear. $\square$

Aside In the Q-polynomial case the actions of A, A^* on E_0V give a special type of tridiagonal pair called a Leonard pair. A Leonard pair is defined as follows.

For the moment let $\mathbb{F}$ denote any field.

Let m = a square matrix over $\mathbb{F}$

Call m tridiagonal whenever each non 0 entry of m is on the diagonal, subdiagonal, or superdiagonal.

Assume m is tridiagonal. Call m irreducible whenever each entry on the subdiagonal is non 0 and each entry on the superdiagonal is non 0.

DEF 24 Given a nonzero, finite-dimensional vectorspace V over $\mathbb{F}$.

A <u>Leonard pair</u> on V is an ordered pair of $\mathbb{F}$-linear maps

$$A : V \to V, \qquad A^* : V \to V$$

st

(i) $\exists$ basis of V wrt which the matrix rep A is irred tridiag and the matrix rep A^* is diagonal.

(ii) $\exists$ basis for V wrt which the matrix rep A^* is irred tridiag and the matrix rep A is diagonal.

Ex Given a nonO, finite-dimensional vector space V over any field $\mathbb{F}$

Given $\mathbb{F}$-linear maps

$$A : V \to V, \qquad A^* : V \to V$$

TFAE

(i) A, A^* is a Leonard pair on V

(ii) A, A^* is a tridiagonal pair on V for which the eigenspaces of A, A^* all have dimension 1.

— o —

The Leonard pairs are classified up to isomorphism.

see

P. Terwilliger. Notes on the Leonard system classification. Graphs Combin. 37 (2021) 1687 – 1748

Back to our DRG $\Gamma = (X, \mathcal{R})$ with diameter $D \geq 1$. $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Fix $x \in X$ and write $T = T(x)$.

COR 25 With reference to Lem 23, assume that the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial. Then the pair A, A^* acts on the primary T-module E_0V as a Leonard pair

pf By Lem 23 and Def 24. $\square$

Exercise IV Given a DRG $\Gamma = (X, R)$ with diameter $D \geq 1$. Fix $x \in X$ and write $T = T(x)$.

Recall the primary T-module $e_0 V$

Show that the following hold on $e_0 V$

- For $0 \leq i \leq D$,

$$E_i^* A E_i^* = a_i E_i^*$$

- Assume the ordering $\{E_i\}_{i=0}^{D}$ is Q-poly. Then for $0 \leq i \leq D$,

$$E_i A^* E_i = a_i^* E_i$$

- For $0 \leq i, j \leq D$

$$E_i^* A^r E_j^* \neq 0 \quad \text{if} \quad r = |i-j|$$

- Assume the ordering $\{E_i\}_{i=0}^{D}$ is Q-poly. Then for $0 \leq i, j \leq D$,

$$E_i A^{*r} E_j \neq 0 \quad \text{if} \quad r = |i-j|$$

Until further notice, assume that Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^{D}$

Our next goal is to show

thm 26 With the above notation, $\exists$ scalars $\omega, \eta, \eta^* \in \mathbb{F}$ such that on E_0V,

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = \gamma^* A^2 + \omega A + \eta I \qquad (AW1)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A = \gamma A^{*2} + \omega A^* + \eta^* I \qquad (AW2)$$

(the scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ are from thm 1 in CH 3)

the AW1, AW2 are called the

<u>Askey-Wilson relations</u>

We will prove thm 26 after a lemma.

Recall the polynomials

$$P(\lambda,\mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda+\mu) - \delta$$
$$P^*(\lambda,\mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma^*(\lambda+\mu) - \delta^*$$

LEM 27 For all $\omega, \eta \in \mathbb{F}$ TFAE

(i) On E_0V,

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^*+A^*A) - \delta A^* \qquad (*)$$
$$= \gamma^*A^2 + \omega A + \eta I$$

(ii) For $0 \le i \le D$,

$$a_i^* P(\theta_i, \theta_i) = \gamma^*\theta_i^2 + \omega\theta_i + \eta$$

pf Let l, r denote LHS, RHS of $(*)$

obs
$$l = \sum_{i=0}^{D}\sum_{j=0}^{D} E_i l E_j$$
$$r = \sum_{i=0}^{D}\sum_{j=0}^{D} E_i r E_j$$

For $0 \le i, j \le D$

$$E_i \ell E_j = E_i A^* E_j \; P(\theta_i, \theta_j)$$

$$E_i r E_j = \delta_{ij} \; E_i \left(\gamma^* \theta_i^2 + \omega \theta_i + \eta \right)$$

(i) → (ii) $\qquad \ell = r$ on $e_0 V$

so for $0 \le i \le D$

$$E_i \ell E_i \overset{\text{on } e_0 V}{=} E_i r E_i$$

$$\| \qquad\qquad\qquad \|$$

$$\underbrace{E_i A^* E_i}_{\| \atop a_i^* E_i} \; P(\theta_i, \theta_i) \qquad\qquad E_i \left(\gamma^* \theta_i^2 + \omega \theta_i + \eta \right)$$

$E_i \neq 0$ on $e_0 V$ so

$$a_i^* P(\theta_i, \theta_i) = \gamma^* \theta_i^2 + \omega \theta_i + \eta$$

(ii) $\to$ (i) Show $\ell = r$ on e_0V

Show for $0 \le i, j \le D$

$$E_i \ell E_j \overset{\text{on } e_0V}{=} E_i r E_j$$

For $0 \le i \le D$

$$E_i \ell E_i = E_i A^* E_i P(\theta_i, \theta_i) \overset{\text{on } e_0V}{=} E_i a_i^* P(\theta_i, \theta_i)$$

$$E_i r E_i = E_i(\gamma^* \theta_i^2 + \omega \theta_i + \eta)$$

So

$$E_i \ell E_i \overset{\text{on } e_0V}{=} E_i r E_i$$

For $0 \le i, j \le D$ with $|i-j| = 1$,

$$E_i \ell E_j = 0 \qquad \text{since} \qquad P(\theta_i, \theta_j) = 0$$

Also $$E_i r E_j = 0 \qquad \text{since} \qquad i \ne j.$$

So $$E_i \ell E_j \overset{\text{on } e_0V}{=} E_i r E_j$$

For $0 \le i, j \le D$ st $|i-j| > 1$,

$$E_i A^* E_j = 0$$

so $E_i \ell E_j = 0$.

Also $E_i r E_j = 0$ since $i \neq j$.

We have shown

$$E_i \ell E_j \overset{\text{on } e_0 V}{=\!=} E_i r E_j \qquad (0 \le i, j \le D)$$

so $\ell = r$ on $e_0 V$ $\square$

the dual Lem 27^* also holds.