

Combinatorics 475

Lecture 30

Stirling numbers

Consider the sequence of polynomials

$$1, \quad x, \quad x^2, \quad x^3, \quad \dots$$

*

Another sequence of polynomials

$$1, \quad x, \quad x(x-1), \quad x(x-1)(x-2), \quad \dots$$

$$\begin{matrix} \text{"} & \text{"} & \text{"} & \text{"} \\ [x]_0 & [x]_1 & [x]_2 & [x]_3 \end{matrix}$$

**

Two problems:

(i) Write * in terms of **

(coefs give Stirling numbers of 2nd kind)

(ii) Write ** in terms of *

(coefs give Stirling numbers of 1st kind)

We start with (i)

Ex Write x^3 in terms of x^k

Sol Consider sequence $\{n^3\}_{n=0}^{\infty}$

Create difference table

0	1	8	27	64	...
1	7	19	37		
	6	12	18		
		6	6		
			0		

So

$$n^3 = 0 \binom{n}{0} + 1 \binom{n}{1} + 6 \binom{n}{2} + 6 \binom{n}{3}$$

$$n^3 = n + 6 \frac{n(n-1)}{2} + 6 \frac{n(n-1)(n-2)}{3 \cdot 2}$$

$$= n + 3n(n-1) + n(n-1)(n-2)$$

$n = 0, 1, 2, \dots$

So

$$x^3 = x + 3x(x-1) + x(x-1)(x-2)$$

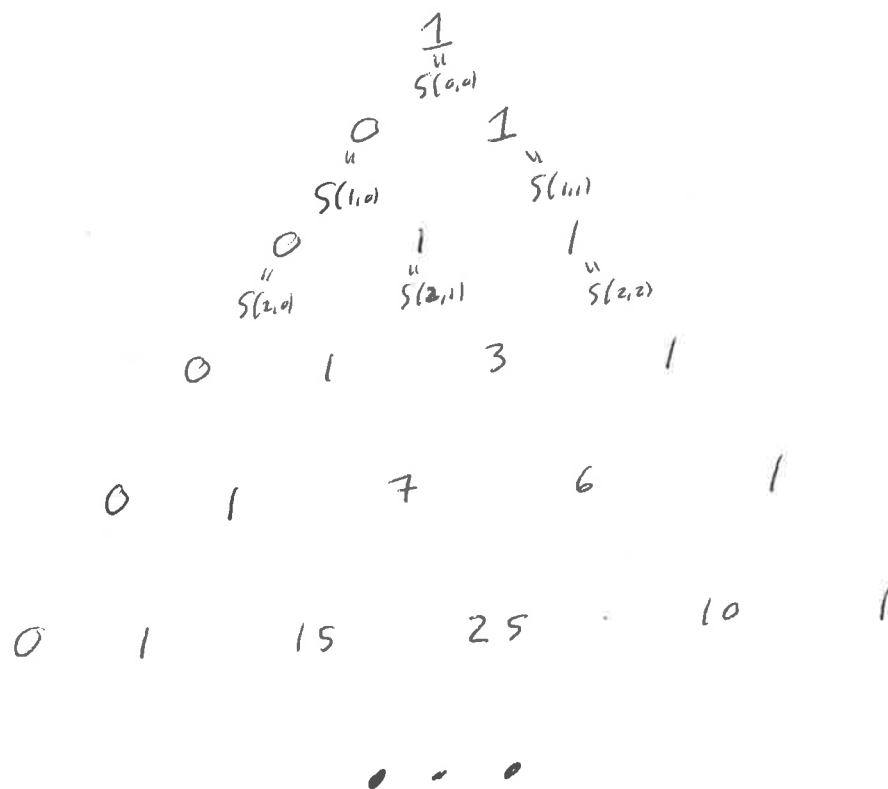
$$= [x]_1 + 3[x]_2 + [x]_3$$

□

More generally we have

	$[x]_0$	$[x]_1$	$[x]_2$	$[x]_3$	$[x]_4$	$[x]_5$
1	1					
x	0	1				
x^2	0	1	1			
x^3	0	1	3	1		
x^4	0	1	7	6	1	
x^5	0	1	15	25	10	1
$\vdots$			...			

View as analog to Pascal triangle :



"Stirling numbers
of 2nd kind"

By construction for $n \geq 0$

$$x^n = \sum_{k=0}^n S(n,k) [x]_k$$

Obs

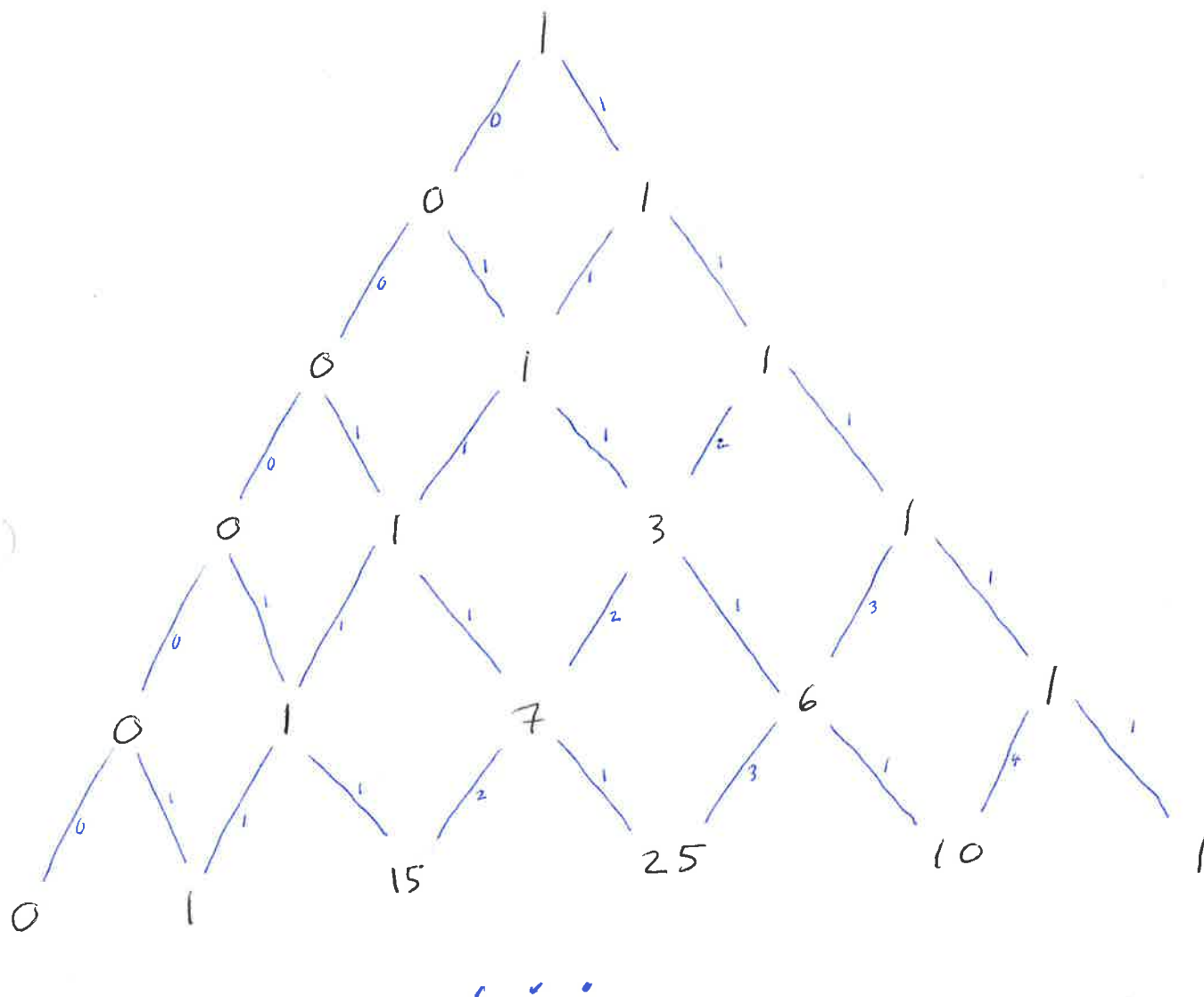
$$S(n,0) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

$$S(n,1) = 1 \quad n \geq 1$$

$$S(n,n) = 1 \quad n \geq 0$$

Next goal: show $S(n, k)$ satisfy a recurrence

Similar to Pascal's formula



For notational convenience define

$$S(n, k) = 0 \quad \text{if } k < 0 \text{ or } k > n$$

thm For $0 \leq k \leq n$

$$S(n, k) = S(n-1, k) + k S(n-1, k-1)$$

pf Assume $n \geq 1$ else true

Recall

$$X^n = \sum_{k=0}^n S(n, k) [x]_k$$

☆

So

$$X^{n+1} = \sum_{k=0}^{n+1} S(n+1, k) [x]_k$$

Also

$$X^n = X^{n+1} X$$

$$= \sum_{k=0}^{n+1} S(n+1, k) [x]_k \underset{\parallel}{X}_{X-k+k}$$

$$X(X-1)(X-2) \dots (X-k+1)$$

$$\left[[x]_k X = [x]_{k+1} + k [x]_k \right]$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} S(n-1, k) [x]_{kn} + \sum_{k=0}^{n-1} k S(n-1, k) [x]_k \\
&\quad \downarrow k \rightarrow kn \\
&\quad \sum_{k=1}^n S(n-1, k-1) [x]_k \\
&\quad \downarrow k=0 \\
&= \sum_{k=0}^n \left(S(n-1, k-1) + k S(n-1, k) \right) [x]_k
\end{aligned}$$

Compare with $\star$ to get

$$S(n, k) = S(n-1, k-1) + k S(n-1, k)$$

$0 \leq k \leq n$

□

$$= \frac{1}{(k-1)!} \sum_{t=1}^k \binom{k-1}{t-1} (-1)^{t-1} (k-t)^{n-1} \quad (t \rightarrow k-1)$$

$$+ \frac{k}{k!} \sum_{t=0}^k \binom{k}{t} (-1)^t (k-t)^{n-1}$$

$$= \frac{k}{k!} \sum_{t=0}^k \left(\binom{k}{t} - \binom{k-1}{t-1} \right) (-1)^t (k-t)^{n-1}$$

" $\binom{k-1}{t}$

$$= \frac{1}{k!} \sum_{t=0}^k \binom{k}{t} (-1)^t (k-t)^n$$

need

$$k \binom{k-1}{t} = \binom{k}{t} (k-t)$$

$$0 \leq t \leq k$$

$$t=k \checkmark$$

$$0=0$$

$$0 \leq t \leq k-1:$$

$$k \frac{(k-1)!}{t! (k-1-t)!} = \frac{k! (k-t)}{t! (k-t)!} \quad \checkmark$$

□

Next goal:

A combinatorial interp of $S(n, k)$

Given sets X, Y with

$$|X| = n, \quad |Y| = k$$

LEM the number of surjective functions $X \rightarrow Y$ is

$$\sum_{t=0}^k \binom{k}{t} (-1)^t (k-t)^n$$

pf. Use incl/excl

Define

$S =$ set of all functions $X \rightarrow Y$

$$|S| = k^n$$

For $1 \leq i \leq k$ define

$A_i =$ set of functions $X \rightarrow Y \setminus \{i\}$

We seek

$$|\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_k|$$

For $\Delta \subseteq \{1, 2, \dots, k\}$

define

$$A_\Delta = \bigcap_{i \in \Delta} A_i$$

= set of functions

$$X \rightarrow (Y \setminus \Delta)$$

Obs

$$\begin{aligned} |A_\Delta| &= |Y \setminus \Delta|^n \\ &= (k - |\Delta|)^n \end{aligned}$$

By incl/excl

$$|\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_k| = \sum_{\Delta \subseteq \{1, 2, \dots, k\}} |A_\Delta| (-1)^{|\Delta|}$$

$$= \sum_{t=0}^k \sum_{\substack{\Delta \subseteq \{1, 2, \dots, k\} \\ |\Delta| = t}} |A_\Delta| (-1)^{|\Delta|}$$

$$= \sum_{t=0}^k \binom{k}{t} (k-t)^n (-1)^t$$

□

COR

 $\forall n \ 0 \leq k \leq n$

$$(i) \ k! S(n, k) = \# \text{ surjective functions} \\ \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, k\}$$

$$(ii) \ k! S(n, k) = \# \text{ ways to put } n \text{ distinguishable objects} \\ \text{into } k \text{ distinguishable boxes, leaving no box empty}$$

$$(iii) \ S(n, k) = \# \text{ ways to put } n \text{ distinguishable objects} \\ \text{into } k \text{ indistinguishable boxes, leaving no box empty}$$

pf By Prev Lem

□

Ex $n = 5$

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$S(n, k) = \# \text{ partitions of } n \text{ distinguishable objects into } k \text{ indistinguishable nonempty boxes}$

k	desc	$S(5, k)$
0	ϕ	0
1	a b c d e	1
2	a b c d (e) a b c (d e)	$5 + \binom{5}{2}$ $= 15$
3	a b c (d) (e) a b (c d) (e)	$\binom{5}{3} + 5 \times 3$ $= 25$
4	a b (c) (d) (e)	$\binom{5}{2} = 10$
5	a (b) (c) (d) (e)	1

