Math 846
Lecture 30

We continue to discuss a DRG $\Gamma = (X, R)$ with $D \geq 1$.

Fix $x \in X$ and write $T = T(x)$

We give more reduction rules associated with the primary T-module.

LEM 10 For $0 \leq i, j \leq D$

(i) $E_0^* A_i E_j^* = \delta_{ij} E_0^* A_i$

(ii) $E_0^* E_i E_j^* = |X|^{-1} m_i u_j(\theta_i) E_0^* A_j$

(iii) $E_0^* A_i A_j^* = m_j u_i(\theta_j) E_0^* A_i$

(iv) $E_0^* E_i A_j^* = \sum_{h=0}^{D} q_{ij}^h E_0^* E_h$

pf Similar to Lem 9. $\square$

We mention some special cases of Lem 9,10.

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LEM 11 For $0 \le j \le D$,

(i) $E_0 E_0^* E_j = |X|^{-1} E_0 A_j^*$

(ii) $E_0 E_0^* A_j = E_0 E_j^*$

(iii) $E_0^* E_0 E_j^* = |X|^{-1} E_0^* A_j$

(iv) $E_0^* E_0 A_j^* = E_0^* E_j$

pf Use Lem 9,10 $\square$

LEM 12

(i) $\quad E_0 E_i^* E_0 = |X|^{-1} k_i E_0 \qquad (0 \le i \le D)$

(ii) $\quad E_0 E_0^* E_0 = |X|^{-1} E_0$

(iii) $\quad E_0^* E_i E_0^* = |X|^{-1} m_i E_0^* \qquad (0 \le i \le D)$

(iv) $\quad E_0^* E_0 E_0^* = |X|^{-1} E_0^*$

pf use Lem 9, 10 $\square$

Note: More reduction rules come by taking the transpose in Lems 9, 10, 11.

Remark By Lem 12,

$$E_i E_0^*, \quad E_0^* E_i, \quad E_i^* E_0, \quad E_0 E_i^*$$

are nonzero for $0 \le i \le D$.

Moreover $\forall B \in M$,

$$B = 0 \text{ iff } BE_0^* = 0 \text{ iff } E_0^* B = 0$$

Also $\forall C \in M^*$,

$$C = 0 \text{ iff } CE_0 = 0 \text{ iff } E_0 C = 0$$

the following reduction rules are of a slightly different nature

LEM 13. For $0 \le i, j \le D$

(i) $A_i E_0^* A_j = |X| E_i^* E_0 E_j^*$

(ii) $E_i E_0^* A_j = A_i^* E_0 E_j^*$

(iii) $A_i E_0^* E_j = E_i^* E_0 A_j^*$

(iv) $E_i E_0^* E_j = |X|^{-1} A_i^* E_0 A_j^*$

pf (i) $A_i E_0^* A_j = |X| A_i E_0^* E_0 E_j^*$ by Lem 10(iii)

$= |X| E_i^* E_0 E_j^*$ by Lem 10(ii) transpose

(ii) - (iv) Similar

$\square$

LEM 14 We have

$$\sum_{i=0}^{D} k_i^{-1} E_i^* E_0 E_i^* = \sum_{j=0}^{D} m_j^{-1} E_j E_0^* E_j \qquad (*)$$

pf $LHS = |X|^{-1} \sum_{i=0}^{D} k_i^{-1} A_i E_0^* A_i$

$$= |X|^{-1} \sum_{i=0}^{D} k_i^{-1} \left(\sum_{r=0}^{D} v_i(\theta_r) E_r \right) E_0^* \left(\sum_{s=0}^{D} v_i(\theta_s) E_s \right)$$

$$= |X|^{-1} \sum_{r=0}^{D} \sum_{s=0}^{D} E_r E_0^* E_s \underbrace{\left(\sum_{i=0}^{D} k_i^{-1} v_i(\theta_r) v_i(\theta_s) \right)}_{= \delta_{rs} m_r^{-1} |X|}$$

$$= \sum_{j=0}^{D} m_j^{-1} E_j E_0^* E_j.$$

$\square$

DEF 15 Define $e_0 = e_0(x)$ to be $|X|$ times the common value of $(*)$.

We observe that $e_0 \in T$ and

$$\overline{e_0} = e_0, \qquad e_0^t = e_0$$

— o —

For the moment, let W denote the primary T-module.

We have

$$V = W + W^\perp \quad \text{(orthog dir sum)}$$

LEM 16 With above notation,

(i) $(e_0 - I)\, W = 0$

(ii) $e_0 W^\perp = 0$

In other words, e_0 acts on V as the orthogonal projection $V \to W$.

pf (i) W has basis $\{\mathbb{1}_i\}_{i=0}^{D}$. Lec 30 8

Fix $0 \le h \le D$,

$$e_0 \mathbb{1}_h = |X| \sum_{i=0}^{D} k_i^{-1} E_i^* E_0 E_i^* \mathbb{1}_h$$

$$= |X| \, k_h^{-1} E_h^* E_0 \underset{\text{``} A_h \hat{x}}{\mathbb{1}_h}$$

$$= |X| \, k_h^{-1} E_h^* \underbrace{V_h(\theta_0)}_{\text{``} k_h} E_0 \hat{x}$$

$$= |X| \, E_h^* E_0 \hat{x}$$

$$= E_h^* J \hat{x}$$

$$= E_h^* \mathbb{1}$$

$$= \mathbb{1}_h$$

(ii) Given $v \in W^{\perp}$ show $e_0 v = 0$.

Obs $e_0 \in T$ and $W^{\perp}$ is T-module so

$$e_0 v \in W^{\perp}$$

By constr $e_0 \in M E_0^* M$

so

$$\begin{aligned} e_0 v &\in M E_0^* M V \\ &\subseteq M E_0^* V \\ &= M \hat{x} \\ &= W. \end{aligned}$$

$E_0^* V = \mathrm{Span}(\hat{x})$

Now $e_0 v \in W \cap W^{\perp} = 0$ $\square$

Recall the center

$$Z(T) = \{ B \in T \mid Bt = tB \ \forall t \in T \}$$

$Z(T)$ is a subalgebra of T.

COR 17 We have

(i) $e_0 \in Z(T)$

(ii) $e_0^2 = e_0$

(iii) $e_0 V$ = primary T-module

(iv) rank $e_0 = D+1$

pf Routine using Lem 16 $\square$

From now on, we denote the primary T-module by $e_0 V$.

Recall $\mathcal{E}_0 V$ has a basis $\{\mathbb{1}_i\}_{i=0}^{p}$ and a basis $\{\mathbb{1}_i^*\}_{i=0}^{p}$

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LEM 18 For $0 \le i, j \le p$

(i) $\langle \mathbb{1}_i, \mathbb{1}_j \rangle = \delta_{ij} k_i$

(ii) $\langle \mathbb{1}_i^*, \mathbb{1}_j^* \rangle = \delta_{ij} |X| m_i$

(iii) $\langle \mathbb{1}_i, \mathbb{1}_j^* \rangle = k_i m_j u_i(\theta_j)$

pf (i) clear

(ii) $LHS = |X|^2 \langle E_i \hat{x}, E_j \hat{x} \rangle$

$$= |X|^2 \langle \hat{x}, E_i E_j \hat{x} \rangle$$

$$= \delta_{ij} |X|^2 \langle \hat{x}, E_i \hat{x} \rangle$$

$$= \delta_{ij} |X|^2 \Big(\underbrace{(x,x)\text{-entry of } E_i}_{= m_i |X|^{-1}} \Big)$$

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(iii) $LHS = |X| \langle A_i \hat{x}, E_j \hat{x} \rangle$

$$= |X| \langle \hat{x}, A_i E_j \hat{x} \rangle$$

$$= |X| v_i(\theta_j) \underbrace{\langle \hat{x}, E_j \hat{x} \rangle}_{= m_j |X|^{-1}}$$

$$= m_j v_i(\theta_j)$$

$$= k_i m_j u_i(\theta_j)$$

$\square$

We now give the action of the T-generators on the basis $\{\mathbb{1}_i\}_{i=0}^{D}$

LEM 19 For $0 \le i,j \le D$

(i) $E_i^* \mathbb{1}_j = \delta_{ij} \mathbb{1}_j$

(ii) $A_i^* \mathbb{1}_j = m_i u_j(\theta_i) \mathbb{1}_j$

(iii) $E_i \mathbb{1}_j = |X|^{-1} m_i k_j u_j(\theta_i) \sum_{h=0}^{D} u_h(\theta_i) \mathbb{1}_h$

(iv) $A_i \mathbb{1}_j = \sum_{h=0}^{D} p_{ij}^h \mathbb{1}_h$

pf Use the reduction rules $\square$

We now give the action of the T-generators on the basis $\{ \mathbb{1}_i^* \}_{i=0}^{D}$

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LEM 20: For $0 \le i, j \le D$

(i) $E_i \mathbb{1}_j^* = \delta_{ij} \mathbb{1}_j^*$

(ii) $A_i \mathbb{1}_j^* = k_i \, u_i(\theta_j) \, \mathbb{1}_j^*$

(iii) $E_i^* \mathbb{1}_j^* = |X|^{-1} k_i m_j \, u_i(\theta_j) \sum_{h=0}^{D} u_i(\theta_h) \, \mathbb{1}_h^*$

(iv) $A_i^* \mathbb{1}_j^* = \sum_{h=0}^{D} q_{ij}^h \, \mathbb{1}_h^*$

pf Similar to the proof of Lem 19 $\square$

Next we show how to transition between

$\{\mathbb{1}_i\}_{i=0}^D$ and $\{\mathbb{1}_i^*\}_{i=0}^D$

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LEM 21 For $0 \le g \le D$,

(i) $$\mathbb{1}_g = |X|^{-1} k_g \sum_{i=0}^{D} u_g(\theta_i)\, \mathbb{1}_i^*$$

(ii) $$\mathbb{1}_g^* = m_g \sum_{i=0}^{D} u_i(\theta_g)\, \mathbb{1}_i$$

pf (i) Use $\mathbb{1}_g = A_g \hat{x}$

$$A_g = k_g \sum_{i=0}^{D} u_g(\theta_i) E_i$$

$$E_i \hat{x} = |X|^{-1} \mathbb{1}_i^* \qquad \forall i$$

(ii) Similar $\square$