

Combinatorics 475

Lecture 2

2.2 Permutations of sets

Given a set S

Given integer $r \geq 0$

An r -permutation of S is a sequence

$$a_1, a_2, \dots, a_r$$

where $a_1, a_2, \dots, a_r$ are distinct elements of S

By a permutation of S we mean an r -perm where $r = |S|$

Ex $S = \{a, b, c\}$

the permutations of S are

abc	cba
bca	acb
cab	bac

the 2-perms of S are

ab	ba	ca
ac	bc	cb

the 1 perms of S are

a	b	c
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No r -perms of S for $r > 3$

Ex Given finite set S write $n = |S|$

For $0 \leq r \leq n$ find the number of r -permutations of S

Sol Select an r -perm $a_1 a_2 \dots a_r$

In stages

stage	to do	# choices
1	pick a_1	n
2	pick a_2	$n-1$
$\vdots$	$\vdots$	$\vdots$
r	pick a_r	$n-r+1$

By multiplication principle

$$\# \text{ } r\text{-perms of } S = \underbrace{n(n-1)(n-2) \dots (n-r+1)}_{\text{call this } P(n, r)}$$

For an integer $n \geq 0$ define

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$

By convention

$$0! = 1$$

Obs

$$P(n, r) = \frac{n!}{(n-r)!}$$

Obs

$$\begin{aligned} \# \text{ perms of } S &= P(n, n) \\ &= n! \end{aligned}$$

"n factorial"

Ex Given set S of people $|S| = n$

For $1 \leq r \leq n$ seat r people from S around circular table. Only care about who sits to your left/right
"circular r -perm"

$$\left[\begin{array}{c} r=4 \\ \begin{array}{ccc} & a & \\ d & & b \\ & c & \end{array} = \begin{array}{ccc} & d & \\ c & & a \\ & b & \end{array} \neq \begin{array}{ccc} & d & \\ a & & c \\ & b & \end{array} \right]$$

How many arrangements possible?

1) Sol (Sim to necklace problem)

Let U = set of all r -perms of S

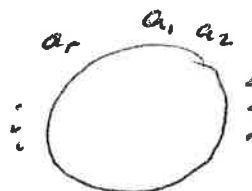
$$|U| = P(n, r)$$

Given r -perm $a_1, a_2, \dots, a_r$ in U

wrap $a_1, a_2, \dots, a_r$ around table clockwise

$a_1, a_2, \dots, a_r$

$\Rightarrow$



the following r -perms get sent to the same table arrangement:

$$a_1 a_2 \dots a_r$$

$$a_2 a_3 \dots a_r a_1$$

$$a_3 \dots a_r a_1 a_2$$

$$\vdots$$

$$a_r a_1 \dots a_{r-1}$$

(*)

Define a binary relation $\sim$ on U as follows:

$$\forall x, y \in U$$

$x \sim y$ whenever x, y give same table arrangement

$\sim$ is equiv relation

Each equiv class has form (*) with r elements

Each table arrangement corresponds to an equiv class $\sim$

$$\begin{aligned} \# \text{ table arrangements} &= \# \text{ equiv classes of } \sim \\ &= \frac{|U|}{\text{card of each equiv class}} \\ &= \frac{P(n, r)}{r} \end{aligned}$$

□

In summary

Then Given a finite set S with $|S|=n$

For $1 \leq r \leq n$ the number of circ r -permutations of S

$$\text{is } \frac{P(n, r)}{r} = \frac{n!}{r (n-r)!}$$

In particular taking $r=n$, the number of circular permutations of S is

$$\frac{n!}{n} = (n-1)!$$

2.3 Combinations of Sets

Given a set S $|S|=n$

For $0 \leq r \leq n$

By an r -combination (or r -subset) of S

we mean a subset of S with cardinality r .

View as selection of r objects from S where selection order does not matter.

Ex Find the number of r -combinations of S .

Sol. Call this number $C(n, r)$

To find it, we count the number of r -perms of S

as follows.

We construct an r -perm from S using this 2-stage procedure:

stage	to do	# choices
1	select an r -combination x from S	$C(n, r)$
2	order the elements of x	$r!$

By multiplication principle

$$\begin{aligned} \# \text{ } r\text{-perms of } S &= C(n, r) r! \\ \text{"} & \\ P(n, r) & \end{aligned}$$

$$\begin{aligned} \text{So } C(n, r) &= \frac{P(n, r)}{r!} \\ &= \frac{n!}{r! (n-r)!} \end{aligned}$$

□

Other notations for $C(n, r)$:

${}_n C_r$

$\binom{n}{r}$

← "binomial coefficient"

↑

we will use

Convention

$$\binom{n}{r} = 0 \quad \text{if} \quad r < 0 \quad \text{or} \quad r > n$$

Note

$$\binom{n}{0} = 1$$

$$\binom{n}{n} = 1$$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$0 \leq r \leq n$$

Recall "Pascal triangle"

lec2 10

the numbers

$$\binom{0}{0}$$

$$\binom{1}{0}$$

$$\binom{1}{1}$$

$$\binom{2}{0}$$

$$\binom{2}{1}$$

$$\binom{2}{2}$$

$\vdots$

are

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$\vdots$

$\vdots$

$\vdots$

thm For $1 \leq k \leq n-1$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

pf 1

Use

formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

pf 2

Consider set

$$S = \{1, 2, \dots, n\}$$

Partition the set of k -subsets of S into 2 types:

type of k -subset of S	# choices
k -subsets of S that contain "1"	$\binom{n-1}{k-1}$ ← # ways to select remaining $k-1$ elements from $\{2, \dots, n\}$
k -subsets of S that do not contain "1"	$\binom{n-1}{k}$ ← # ways to select k elements from $\{2, 3, \dots, n\}$

So by addition principle

$$\begin{aligned} \binom{n}{k} &= \# \text{ } k\text{-subsets of } S \\ &= \binom{n-1}{k-1} + \binom{n-1}{k} \end{aligned}$$

□

Here is another formula for binom coefficients

Lec2

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thm For an integer $n \geq 0$

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

pf. Consider the set

$$S = \{1, 2, \dots, n\}$$

$$2^n = \# \text{ subsets of } S$$

$$= \# \text{ 0-subsets of } S$$

$$+ \# \text{ 1-subsets of } S$$

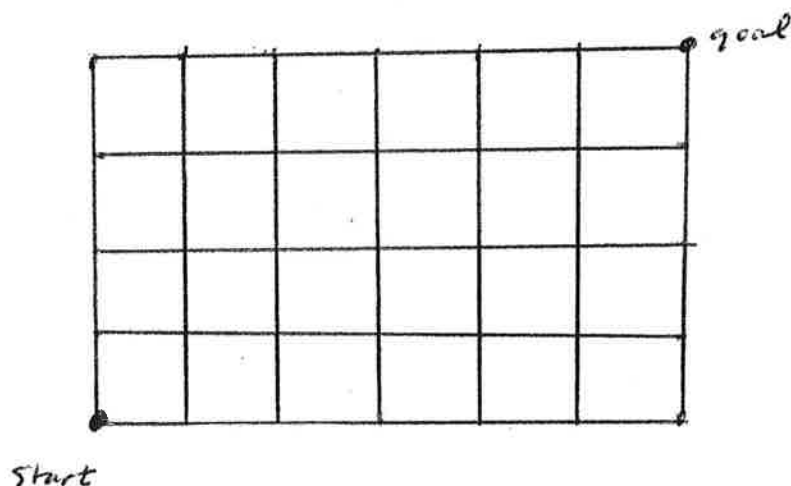
$$+ \vdots$$

$$+ \# \text{ n-subsets of } S$$

$$= \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

□

Ex Consider 4×6 grid of city streets!



goal is 6 blocks East
4 blocks North

Go from "start" to "goal" in 10 steps, each N or E

How many paths?

Sol

Lec2 14

Consider 10 steps of path

1 2 3 4 5 6 7 8 9 10

↑ ↑ ...

E_nN E_nN

Fill each blank with "E" or "N"

Require 6 E's and 4 N's

For each path,

the set of blanks labelled N is a 4-subset of $\{1, 2, \dots, 10\}$

set of paths $\overset{\text{bij}}{\iff}$ 4-subsets of $\{1, 2, \dots, 10\}$

$$\# \text{ paths} = \# \text{ 4-subsets of } \{1, 2, \dots, 10\}$$

$$= \binom{10}{4}$$

□

