Math 846

Lecture 2

We continue to discuss a graph $\Gamma = (X, \mathcal{R})$

with spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_0 & m_1 & \cdots & m_r \end{pmatrix}$$

COR 2 We have

(i) $$\sum_{i=0}^{r} m_i = |X|$$

(ii) $$\sum_{i=0}^{r} \theta_i m_i = 0$$

(iii) $$\sum_{i=0}^{r} \theta_i^2 m_i = 2|\mathcal{R}|$$

pf. Set $\ell = 0, 1, 2$ in LEM 1 $\square$

A <u>bipartition</u> of Γ is a partition

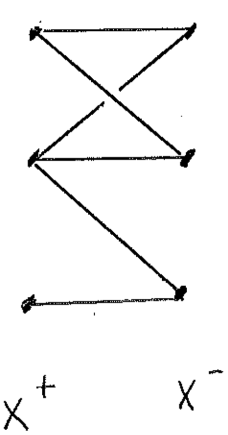
$$X = X^+ \cup X^- \qquad \text{(disjoint union)}$$

such that

X^+ and X^- contain no edges.

Γ is <u>bipartite</u> whenever Γ has a bipartition.

Ex A bipartite graph:



One checks that Γ is bipartite iff

Γ has no closed walks of odd length (ex)

Our next goal is to prove the following result

LEM 3 For the graph $\Gamma = (X, \mathcal{R})$ TFAE:

(i) Γ is bipartite

(ii) For $0 \le i \le r$, both

$$\theta_i = -\theta_{r-i},$$

$$m_i = m_{r-i}$$

where $\theta_0 > \theta_1 > \cdots > \theta_r$

To prove LEM 3 we use the following result.

LEM 4 Given a function

$$m : \mathbb{F} \to \mathbb{N}$$

such that

$$m(\lambda) \neq 0 \quad \text{for finitely many } \lambda \in \mathbb{F}.$$

Then TFAE:

(i) $m(\lambda) = m(-\lambda) \quad \forall\ 0 \neq \lambda \in \mathbb{F}$

(ii) For all odd $\ell \in \mathbb{N}$,

$$\sum_{\lambda \in \mathbb{F}} m(\lambda)\,\lambda^{\ell} = 0$$

pf (i) → (ii) Clear

(ii) → (i) Define a function

$$\mu : \mathbb{F} \to \mathbb{N}$$

such that

$$\mu(0) = m(0)$$

$$\mu(\lambda) = \min\{m(\lambda), m(-\lambda)\} \qquad 0 \neq \lambda \in \mathbb{F}$$

By construction

$$\mu(\lambda) = \mu(-\lambda) \qquad 0 \neq \lambda \in \mathbb{F}$$

Show $\mu = m$

Define $\Delta = m - \mu$

Show $\Delta = 0$

By constr

$$\Delta(0) = 0$$

$$\Delta(\lambda) = 0 \quad \text{or} \quad \Delta(-\lambda) = 0 \qquad 0 \neq \lambda \in \mathbb{F}$$

For odd $\ell \in \mathbb{N}$

$$\sum_{\lambda \in \mathbb{F}} \Delta(\lambda)\lambda^\ell = \underbrace{\sum_{\lambda \in \mathbb{F}} m(\lambda)\lambda^\ell}_{= 0} - \underbrace{\sum_{\lambda \in \mathbb{F}} \mu(\lambda)\lambda^\ell}_{= 0}$$

So

$$\sum_{\lambda \in \mathbb{F}} \Delta(\lambda)\lambda^{\ell} = 0 \qquad (*)$$

To show $\Delta = 0$, we define a set

$$S = \{ \lambda \in \mathbb{F} \mid \Delta(\lambda) \neq 0 \}$$

and show $S = \phi$.

By constr

$$0 \notin S$$

$$\lambda \in S \quad \text{implies} \quad -\lambda \notin S \qquad 0 \neq \lambda \in \mathbb{F}$$

Suppose $S \neq \phi$.

Write $S = \{ \lambda_0, \lambda_1, \dots, \lambda_N \}$ $\qquad N \in \mathbb{N}$

observe $\lambda_0^2, \lambda_1^2, \dots, \lambda_N^2$ mutually distinct.

By $(*)$,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0^2 & \lambda_1^2 & \cdots & \lambda_N^2 \\ \lambda_0^4 & \lambda_1^4 & \cdots & \lambda_N^4 \\ \vdots & & & \vdots \\ \lambda_0^{2N} & \lambda_1^{2N} & \cdots & \lambda_N^{2N} \end{pmatrix} \begin{pmatrix} \Delta(\lambda_0)\lambda_0 \\ \Delta(\lambda_1)\lambda_1 \\ \vdots \\ \Delta(\lambda_N)\lambda_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

matrix is Vandermonde, hence invertible

must be 0 vector

So

$$\underset{0}{\underset{\#}{\Delta(\lambda_i)}}\ \underset{0}{\underset{\#}{\lambda_i}} = 0 \qquad 0 \le i \le N$$

Cont

Therefore, $S = \phi$ so $\Delta = 0$

$\square$

Proof of LEM 3

Γ is bipartite

iff

Γ has no closed walks of odd length

iff

$$\sum_{i=0}^{r} m_i \theta_i^{\ell} = 0 \qquad \forall \text{ odd } \ell \in \mathbb{N}$$

Result follows by LEM 4 using

$$m(\lambda) = \begin{cases} m_i & \text{if } \lambda = \theta_i \text{ is an eigenvalue of } \Gamma \\ 0 & \text{if } \lambda \text{ is not an eigenvalue of } \Gamma \end{cases} \qquad \lambda \in \mathbb{F}$$

$\square$

DEF 5 The <u>adjacency algebra</u> of Γ is the subalgebra M of $Mat_X(\mathbb{F})$ generated by the adjacency matrix A.

For $x, y \in X$ the <u>distance</u>

$$\partial(x,y) = \min\{\ell \mid \exists \text{ walk of length } \ell \text{ from } x \text{ to } y\}$$

$$\in \mathbb{N} \cup \infty$$

Γ is <u>connected</u> whenever $\partial(x,y) < \infty \quad \forall x, y \in X$

Until further notice, assume Γ is connected.

For $x \in X$ define

$$d(x) = \max\{\partial(x,y) \mid y \in X\}$$ "diameter wrt x"

Define $$D = \max\{d(x) \mid x \in X\}$$ "diameter of Γ"

LEM 6 A connected graph $\Gamma = (X, R)$ with diameter D has at least $D+1$ dist eigenvalues.

pf the adjacency algebra M has a basis $\{A^i\}_{i=0}^{r}$

where $r+1$ = # distinct eigvalues of Γ.

show $r \geq D$

Suppose $r < D$

$\exists$ $x, y \in X$ with $\partial(x,y) = r+1$

For $0 \leq i \leq r+1$

$$(A^i)_{xy} = \text{\# walks of length } i \text{ from } x \text{ to } y$$

$$= \begin{cases} 0 & \text{if } 0 \leq i \leq r \\ \neq 0 & \text{if } i = r+1 \end{cases}$$

So $\quad B_{xy} = 0 \quad \forall B \in M$

and $\quad (A^{r+1})_{xy} \neq 0 \qquad A^{r+1} \in M$

Contr.

So $\quad r \geq D$. $\square$

Assume Γ is connected

Until further notice, fix $x \in X$ "base vertex"

Write $d = d(x)$

For $i \in \mathbb{N}$ define
$$\Gamma_i(x) = \{ y \in X \mid \partial(x,y) = i \}$$

So
$$\Gamma_i(x) \neq \phi \qquad (0 \le i \le d)$$

$$\Gamma_0(x) = \{x\}, \qquad \Gamma_1(x) = \Gamma(x)$$

For $0 \le i \le d$ define a diagonal matrix
$$E_i^* = E_i^*(x) \in \mathrm{Mat}_X(\mathbb{F})$$

with (y,y)-entry
$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } y \in \Gamma_i(x) \\ 0 & \text{if } y \notin \Gamma_i(x) \end{cases} \qquad y \in X$$

Obs
$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \le i,j \le d)$$
$$I = \sum_{i=0}^{d} E_i^*$$

$$E_i^* V = \text{Span}\left(\hat{y} \mid y \in \Gamma_i(x) \right)$$

"ith subconstituent of Γ with respect to x"

$$V = \sum_{i=0}^{d} E_i^* V \quad \text{(orthog direct sum)}$$

By the above comments, $\{E_i^*\}_{i=0}^{d}$ form a basis

for a commutative subalgebra

$$M^* = M^*(x) \subseteq \text{Mat}_X(\mathbb{F})$$

We call M^* the

<u>dual adjacency algebra of Γ with respect to x</u>

We call E_i^* the

<u>ith dual primitive idempotent of Γ with respect to x</u>

DEF 7 With above notation,
let $T = T(x)$ denote the subalgebra
of $Mat_X(\mathbb{F})$ generated by M and M^*.

We call T the
<u>subconstituent algebra of Γ with respect to x</u>