

Combinatorics 475

Lecture 29

Next general goal:

Given $h = \{h_n\}_{n=0}^{\infty} \in \mathbb{V}$

Suppose there exists $p \geq 0$ such that

$$\Delta^{p+1} h = 0$$

Show there exists a polynomial $f(x)$ of degree $\leq p$ such that

$$h_n = f(n) \quad \text{for } n \geq 0$$

Given a sequence $\{h_n\}_{n=0}^{\infty}$

Construct difference table

h_0	h_1	h_2	h_3	h_4	...
Δh_0	Δh_1	Δh_2	Δh_3	...	
$\Delta^2 h_0$	$\Delta^2 h_1$	$\Delta^2 h_2$	...		
	$\Delta^3 h_0$	$\Delta^3 h_1$	...		
		$\Delta^4 h_0$	...		

diagonal sequence

the corresp diagonal sequence is

$$\{\Delta^n h_0\}_{n=0}^{\infty}$$

Ex Define

$$h_n = \binom{n}{3} \quad n = 0, 1, 2, \dots$$

Find the corresp diagonal sequence

Sol Create difference table

$\binom{0}{3}$ 0	$\binom{1}{3}$ 0	$\binom{2}{3}$ 0	$\binom{3}{3}$ 1	$\binom{4}{3}$ 4	$\binom{5}{3}$ 10	$\binom{6}{3}$ 20	...
	0	0	1	3	6	10	...
		0	1	2	3	4	...
			1	1	1	1	...
				0	0	0	...

Diagonal sequence is

0, 0, 0, 1, 0, 0, ...

LEM Fix an integer $p \geq 0$

For the sequence

$$\left\{ \binom{n}{p} \right\}_{n=0}^{\infty}$$

The corresp diagonal sequence is

$$0, 0, \dots, 0, 1, 0, 0, \dots$$

↑

Location p

pf

Denote the diagonal sequence by $\{c_n\}_{n=0}^{\infty}$

show

$$c_n = \begin{cases} 1 & \text{if } n=p \\ 0 & \text{if } n \neq p \end{cases} \quad n \geq 0$$

Create difference table

$$\begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ p \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ p \\ 0 \end{pmatrix} \quad \dots \quad \begin{pmatrix} p-1 \\ p \\ 0 \end{pmatrix} \quad \begin{pmatrix} p \\ p \\ 1 \end{pmatrix} \quad \begin{pmatrix} p+1 \\ p \\ p+1 \end{pmatrix} \quad \dots$$

$$0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 1 \quad p \quad \dots$$

$$0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 1 \quad p^2 \quad \dots$$

$$0 \quad 0 \quad \dots \quad 0 \quad 1 \quad p^3 \quad \dots$$

...

$$0 \quad 1 \quad 2 \quad \dots$$

$$1 \quad 1 \quad \dots$$

$$0 \quad \dots$$

row p

*

$$\text{So } c_n = 0 \quad 0 \leq n \leq p-1$$

$$\text{and } c_p = 1$$

$$\text{Recall } \binom{n}{p} = \frac{n(n-1)(n-2) \dots (n-p+1)}{p!} = \text{poly in } n \text{ with degree } p$$

So by prev thm

$$\Delta^{p+1} \text{ sends } \left\{ \binom{n}{p} \right\}_{n=0}^{\infty} \text{ to } \textcircled{0}$$

Therefore

$$c_n = 0 \quad \forall \quad n \geq p+1$$

□

Cor. A sequence $\{h_n\}_{n=0}^{\infty}$ is uniquely determined by its diagonal sequence

pf. Suppose $\{h_n\}_{n=0}^{\infty}$ and $\{h'_n\}_{n=0}^{\infty}$ have the same diagonal sequence. Show $h_n = h'_n$ $n = 0, 1, 2, \dots$

Then the sequence $\{h_n - h'_n\}_{n=0}^{\infty}$ has diagonal sequence $0, 0, 0, \dots$

Now $\{h_n - h'_n\}_{n=0}^{\infty} = \textcircled{1}$

so $h_n = h'_n$ $n = 0, 1, 2, \dots$

□

Ex Find the sequence $\{h_n\}_{n=0}^{\infty}$ whose diagonal sequence is

$$1, 3, 2, 5, 0, 0, \dots$$

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Sol. View * as

$$\begin{array}{rcl}
 & 1 \text{ times} & 1, 0, 0, 0, \dots \\
 + & & \\
 & 3 \text{ times} & 0, 1, 0, 0, \dots \\
 + & & \\
 & 2 \text{ times} & 0, 0, 1, 0, \dots \\
 + & & \\
 & 5 \text{ times} & 0, 0, 0, 1, \dots
 \end{array}$$

So $h_n = 1 \binom{n}{0} + 3 \binom{n}{1} + 2 \binom{n}{2} + 5 \binom{n}{3} \quad n \geq 0$

thm Fix integer $p \geq 0$

Given real numbers

$$c_0, c_1, \dots, c_p$$

Let $\{h_n\}_{n=0}^{\infty}$ denote the sequence with diagonal sequence

$$c_0, c_1, c_2, \dots, c_p, 0, 0, 0, \dots$$

then

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + c_2 \binom{n}{2} + \dots + c_p \binom{n}{p}$$

pf clear from prev Lem + Cor

□

thm Given sequence $h = \{h_n\}_{n=0}^{\infty}$

Assume $\exists$ integer $p \geq 0$ such that

$$\Delta^{p+1} h = 0$$

then $\exists$ polynomial $f(x)$ of degree $\leq p$ such that

$$h_n = f(n) \quad n = 0, 1, 2, \dots$$

pf Let $\{c_n\}_{n=0}^{\infty}$ denote the diagonal sequence for h .

Obs $c_n = 0$ if $n \geq p+1$

By prev thm

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \dots + c_p \binom{n}{p} \quad n = 0, 1, 2, \dots$$

Define

$$f(x) = c_0 \binom{x}{0} + c_1 \binom{x}{1} + \dots + c_p \binom{x}{p}$$

$$f(x) \text{ has degree } \begin{cases} p & \text{if } c_p \neq 0 \\ < p & \text{if } c_p = 0 \end{cases}$$

$$h_n = f(n) \quad n = 0, 1, 2, \dots$$

□

Ex Find a formula for

$$1^4 + 2^4 + 3^4 + \dots + n^4$$

$$n \geq 0$$

Sol. Consider sequence $\{n^4\}_{n=0}^{\infty}$

Find the diagonal sequence

0^4 0	1^4 1	2^4 16	3^4 81	4^4 256	5^4 625
	1	15	65	175	369
		14	50	110	194
			36	60	84
				24	24
					0

Diag sequence is

$$0, 1, 14, 36, 24, 0, 0, \dots$$

So

$$n^4 = \binom{n}{1} + 14 \binom{n}{2} + 36 \binom{n}{3} + 24 \binom{n}{4}$$

Recall

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$$

$$n, k \geq 0$$

So

$$1^4 + 2^4 + 3^4 + \dots + n^4 =$$

$$\binom{n+1}{2} + 14 \binom{n+1}{3} + 36 \binom{n+1}{4} + 24 \binom{n+1}{5}$$

Ex Given a polynomial $f(x)$ deg p

Find formula for

$$f(0) + f(1) + \dots + f(n)$$

$$n = 0, 1, 2, \dots$$

Sol. Consider sequence $\{f(n)\}_{n=0}^{\infty}$

Let $\{c_n\}_{n=0}^{\infty}$ denote corresp diagonal sequence.

Obs

$$c_n = 0 \quad \text{if } n > p$$

$$f(n) = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \dots + c_p \binom{n}{p}$$

$$n \geq 0$$

Now

$$f(0) + f(1) + \dots + f(n) =$$

$$c_0 \binom{n+1}{0} + c_1 \binom{n+1}{1} + \dots + c_p \binom{n+1}{p}$$

$$n = 0, 1, 2, \dots$$

□

