Math 846

Lecture 29

We continue to discuss a DRG $\Gamma = (X, R)$ with diam $D \geq 1$. Fix $x \in X$ and write $T = T(x)$

COR 4 Assume the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial. Then the pair A, A^* acts on each irreducible T-module W as a tridiagonal pair in the sense of Def 20 (Ch 1)

pf By Lems 1–3. $\square$

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Prop 5 Let W, U denote nonisomorphic irred T-modules. Then W, U are orthogonal.

pf Suppose W, U are not orthog. We display a T-module iso $W \to U$.

By constr
$$V = U + U^{\perp} \quad \text{(orthog dir sum)}$$

Given $v \in V$ $\exists$ unique $u \in U$ and $u' \in U^{\perp}$ s.t
$$v = u + u'$$

Call u the orthogonal projection of v onto U.

$\forall w \in W$ let $\sigma(w)$ denote the orthogonal projection of w onto U. So $\sigma(w)$ is the unique vector in V s.t
$$\sigma(w) \in U, \qquad w - \sigma(w) \in U^{\perp}.$$

Show
$$\sigma : \begin{array}{l} W \to U \\ w \to \sigma(w) \end{array}$$

is a T-module iso.

<u>claim 1</u> $\quad (B\sigma - \sigma B)|W = 0 \qquad \forall B \in T$

<u>pf cl</u> $\quad \forall w \in W,$

$$B\sigma(w) \in U \qquad \text{since } U \text{ is } T\text{-module}$$

and

$$Bw - B\sigma(w) \in U^{\perp} \qquad \text{since } U^{\perp} \text{ is } T\text{-module}$$

so

$$B\sigma(w) = \sigma(Bw)$$

Claim proved ✓

<u>claim 2</u> $\quad \sigma : W \to U$ is injective

<u>pf cl</u> $\quad$ let K = kernel of σ on W.

By claim 1, $\quad K$ is T-submodule of W.

By the irred of W,

$$K = 0 \quad \text{or} \quad K = W$$

$K \neq W$ since $\langle W, U \rangle \neq 0$, so $K = 0$

claim proved ✓

Claim 3 $\sigma : W \to U$ is surjective.

pf of Let Im denote the image of σ in U.

By claim 1,

Im is a T-submodule of U.

By the irred of U,

$Im = 0$ or $Im = U$.

But $Im \neq 0$ since $\langle w, u \rangle \neq 0$, so

$Im = U$.

claim proved ✓.

We have shown $\sigma : W \to U$ is a T-module iso.

Result follows. $\square$

<u>Notation</u> Let $\Psi = \Psi(x)$ denote the set of isomorphism classes of irred T-modules.

The elements of Ψ are called <u>types</u>.

For $\phi \in \Psi$ define

V_ϕ = subspace of V spanned by the irred T-modules of type ϕ.

Call V_ϕ the <u>ϕ-homogeneous component</u> of V.

Observe that V_ϕ is a T-module.

By Prop 5,

$$V = \sum_{\phi \in \Psi} V_\phi \quad \text{(orthog dir sum of } T\text{-modules)}$$

Given $\phi \in \Psi$ and an irreducible T-module $W \subseteq V_\phi$ the dimension, diameter, etc of W depends only on ϕ.

So we often denote these by

$\dim \phi$, $d(\phi)$, etc

Write V_ϕ as an orthogonal direct sum of irreducible T-modules:

$$V_\phi = W_1 + W_2 + \cdots + W_m \qquad (*)$$

[this decomp is not unique]

For $1 \le i \le m$ W_i has type ϕ so

$$\dim(V_\phi) = m \dim(\phi)$$

so

$$m = \frac{\dim(V_\phi)}{\dim(\phi)}$$

In particular m is indep of which decomp $(*)$ is used.

We call m the <u>multiplicity of ϕ in V</u>

We now consider a certain irred T-module called the primary T-module. Recall

$$\mathbb{1} = \sum_{y \in X} \hat{y}$$

For $0 \le i \le D$ define

$$\mathbb{1}_i = \sum_{y \in \Gamma_i(x)} \hat{y}$$

obs

$$\mathbb{1}_i = E_i^* \mathbb{1}$$
$$= A_i \hat{x}$$

LEM 6 For $0 \le j \le D$,

$$|X| \, E_j \hat{x} = A_j^* \mathbb{1}$$

(call this vector $\mathbb{1}_j^*$)

pf Recall

$$A_j^* \mathbb{1} = |X| \, E_j \hat{x} \circ \mathbb{1}$$

$$= |X| \, E_j \hat{x} \qquad \square$$

LEM 7 The vector space $M\hat{x} = M^* \mathbb{1}$ is an irreducible T-module with bases $\{\mathbb{1}_i\}_{i=0}^{D}$ and bases $\{\mathbb{1}_i^*\}_{i=0}^{D}$. This T-module has endpt 0, dual endpt 0, diameter D, dual diameter D.

This T-module is thin and dual thin.

pf We have $M\hat{x} = M^* \mathbb{1}$ by Lem 6, call this space W.

Obs $\quad MW \subseteq W$ since $W = M\hat{x}$

$\quad M^*W \subseteq W$ since $W = M^*\mathbb{1}$

So W is T-module.

Suppose T-module W is reducible. Then W is orthog dir sum of irred T-modules.

Since $\hat{x} \in W$ these irred T-modules are not all orthog to $\hat{x}$.

So one of them has endpt 0 and hence contains $\hat{x}$.

But then it contains $M\hat{x} = W$. So it equals W.

Concerning the remaining assertions, for $0 \le i \le D$

$$E_i^* W = E_i^* M^* \mathbb{1}$$
$$= \operatorname{Span}(E_i^* \mathbb{1}) \qquad \text{has dim } 1$$

$$E_i W = E_i M \hat{x}$$
$$= \operatorname{span}(E_i \hat{x}) \qquad \text{has dim } 1 \qquad \square$$

DEF 8 The T-module

$$M\hat{x} = M^* \mathbb{1}$$

is called <u>primary</u> (or <u>trivial</u>)

We next do a careful study of the primary T-module.
To this end, we first obtain some "reduction rules"
involving E_0 and E_0^*,

LEM 9 For $0 \le i, j \le D$,

(i) $E_0 A_i^* E_j = \delta_{ij} E_0 A_i^*$

(ii) $E_0 E_i^* E_j = |X|^{-1} k_i u_i(\theta_j) E_0 A_j^*$

(iii) $E_0 A_i^* A_j = k_j u_j(\theta_i) E_0 A_i^*$

(iv) $E_0 E_i^* A_j = \sum_{h=0}^{D} p_{ij}^h E_0 E_h^*$

pf (i) Recall

$$E_0 A_i^* E_j = 0 \text{ iff } q_{ij}^0 = 0$$

If $i \neq j$ then $q_{ij}^0 = 0$ so $E_0 A_i^* E_j = 0$

Also
$$E_0 A_i^* = E_0 A_i^* (E_0 + \cdots + E_D)$$
$$= E_0 A_i^* E_i$$

(ii) Use
$$E_i^* = |X|^{-1} \sum_{h=0}^{D} v_i(\theta_h) A_h^*$$

and (i) above.

(iii) Use

$$A_j = \sum_{h=0}^{D} v_j(\theta_h) E_h$$

and (i) above.

(iv) Recall by LEM 57 in CH1

$$E_0 E_0^* \beta = E_0 \beta \rho \qquad \forall \beta \in M$$

so

$$E_0 E_0^* A_j = E_0 E_j^*$$

Now

$$E_0 E_i^* A_j = E_0 E_0^* A_i A_j$$

$$= \sum_{h=0}^{D} p_{ij}^h E_0 E_0^* A_h$$

$$= \sum_{h=0}^{D} p_{ij}^h E_0 E_h^*$$

$\square$