

Combinatorics 475

Lecture 28

8.1 cont

Recall n th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$n = 0, 1, 2, \dots$

C_n and Multiplication schemes

Given n variables, say $n=4$

a, b, c, d

We can compute the product $abcd$ in the following ways

$((a b) c) d$ $((a b)(c d))$ $((a(b c)) d)$ $(a((b c) d))$ $(a(b(c d)))$

Call these multiplication schemes

Let $h_n = \# \text{ mult schemes involving } n \text{ variables}$

Find h_n
One checks

n	1	2	3	4	5	...
h_n	1	1	2	5	14	...

So it appears

$$h_n = C_{n-2} \quad n = 1, 2, \dots$$

Thm For $n \geq 1$

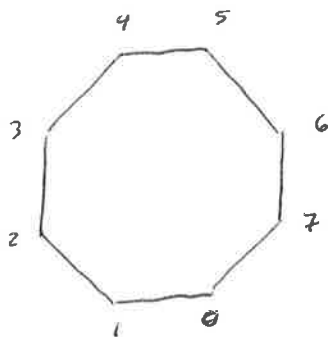
$C_{n-2} = \#$ of multiplication schemes involving n variables

pf Call the variables $a_1, a_2, \dots, a_n$

Consider a convex $(n+1)$ -gon P .

ex $n=7$

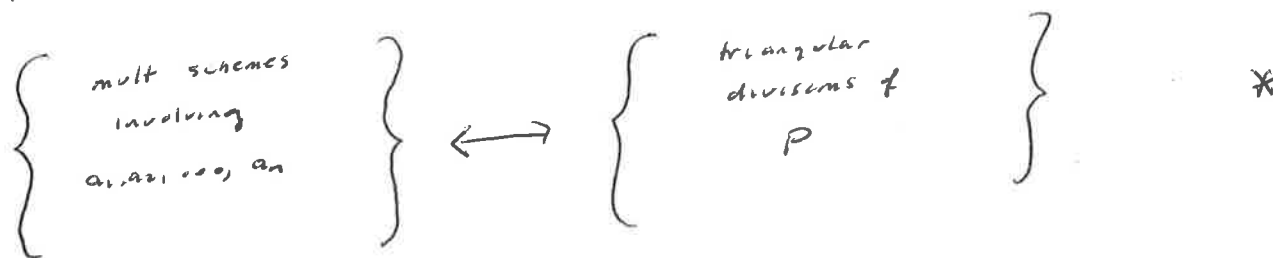
P :



Recall from Section 7.6

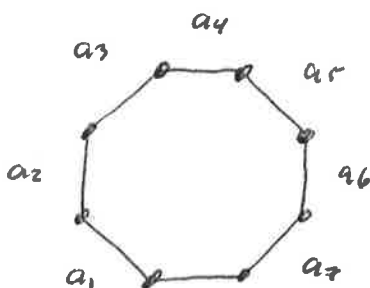
$C_{n-2} = \#$ triangular divisions of P

We display a bijection



For $1 \leq i \leq n$ give the edge i -th of P the label a_i

ex $n=7$

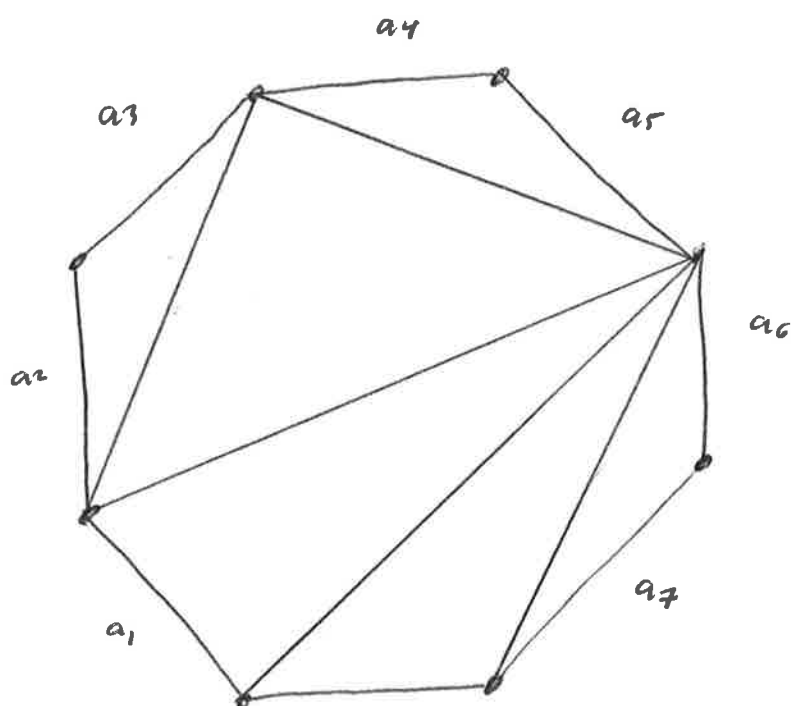


Given a mult scheme involving $a_1, a_2, \dots, a_n$

ex $n=7$

$$\left(\left(a_1 \left(\left(a_2 a_3 \right) \left(a_4 a_5 \right) \right) \right) \left(a_6 a_7 \right) \right)$$

Use this to divide P :

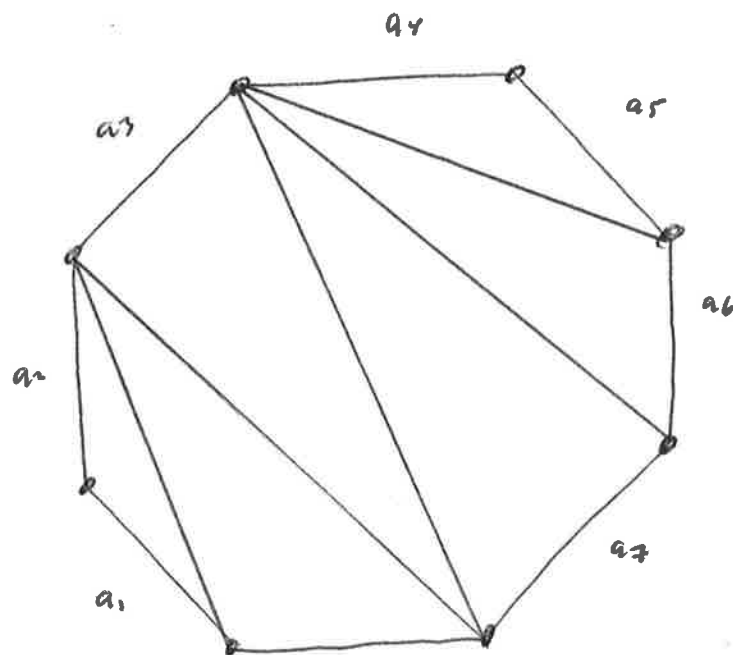


this gives triangular division of P

Division process is reversible:

Given triangular division of P , find corresp mult scheme
for $a_1, a_2, \dots, a_n$

ex $n=7$



yields

$$((a_1 \ a_2) (a_3 (((a_4 \ a_5) a_6) a_7))))$$

We have displayed a bijection *

Result follows.

□

8.2 Difference Sequences and Stirling numbers

Given a sequence of real numbers $\{h_n\}_{n=0}^{\infty}$

Define a new sequence $\{\Delta h_n\}_{n=0}^{\infty}$ where

$$\Delta h_n = h_{n+1} - h_n$$

View Δ as an operator that acts on all sequences $\{h_n\}_{n=0}^{\infty}$

Ex take $h_n = n^2$

$$\{n^2\}_{n=0}^{\infty}$$

Δ

$\downarrow$

$$\{2n+1\}_{n=0}^{\infty}$$

$$(n+1)^2 - n^2 = 2n+1$$

Δ

$\downarrow$

$$\{2\}_{n=0}^{\infty}$$

$$2(n+1)+1 - (2n+1) = 2$$

Δ

$\downarrow$

$$\{0\}_{n=0}^{\infty}$$

$$2 - 2 = 0$$

Define

$$\mathbb{Q} = \{0\}_{n=0}^{\infty} \\ = (0, 0, 0, \dots)$$

For $r \geq 0$ the operator Δ^r acts by applying Δ r times.

Δ^0 means "do nothing"

$$\Delta^0 h_n = h_n$$

$$n = 0, 1, 2, \dots$$

Δ^r sends $\{n^2\}_{n=0}^{\infty}$ to $\mathbb{Q}$ for $r \geq 3$

View

	0	1	4	9	16	25	...
$\Delta \downarrow$		1	3	5	7	9	...
$\Delta \downarrow$			2	2	2	2	...
$\Delta \downarrow$				0	0	0	...
$\Delta \downarrow$					0	0	...
$\vdots$							

"difference table"

Define

$V =$ set of all sequences of real numbers $\{h_n\}_{n=0}^{\infty}$

Given $h, h' \in V$

Write

$$h = \{h_n\}_{n=0}^{\infty}$$

$$h' = \{h'_n\}_{n=0}^{\infty}$$

Define

$$h+h' = \{h_n+h'_n\}_{n=0}^{\infty} \in V$$

Given a real number α

$$\text{Define } \alpha h = \{\alpha h_n\}_{n=0}^{\infty} \in V$$

Obs

$$\alpha(h+h') = \alpha h + \alpha h'$$

$$h + 0 = 0 + h = h$$

V is a "vector space"

the function

$$\Delta: V \rightarrow V$$

respects addition and scalar mult as follows:

$$\Delta(h+h') = \Delta h + \Delta h'$$

$$\forall h, h' \in V$$

$$\Delta(\alpha h) = \alpha \Delta h$$

$$\forall \alpha \in \mathbb{R} \quad \forall h \in V$$

" Δ is a linear transformation of V "

— o —

Question

what elements of V does Δ^r
send to 0 for suf large r ?

Thm Given any polynomial $f(x)$

let $p = \text{degree of } f$

Consider the sequence

$$\{f(n)\}_{n=0}^{\infty}$$

Then

Δ^{p+1} sends this sequence to $\textcircled{0}$

pf Use induction on p

Case $p=0$: true ✓

Case $p \geq 1$:

Write

$$f(x) = a_0 + a_1 x + \dots + a_p x^p$$

$$\begin{aligned} \text{So } f(n) &= a_0 + a_1 n + a_2 n^2 + \dots + a_p n^p \\ &= \sum_{k=0}^p a_k n^k \end{aligned}$$

$$\text{So } \Delta^{p+1} \{f(n)\}_{n=0}^{\infty} = \sum_{k=0}^p a_k \Delta^{p+1} \{n^k\}_{n=0}^{\infty}$$

Suf to show

$$\Delta^{p+1} \{n^k\}_{n=0}^{\infty} = \textcircled{0} \quad 0 \leq k \leq p$$

Suf to show

$$\Delta^{p+1} \{n^p\}_{n=0}^{\infty} = \textcircled{0}$$

Write

$$\Delta \{n^p\}_{n=0}^{\infty} = \{h_n\}_{n=0}^{\infty}$$

$$h_n = (n+1)^p - n^p$$

$$= \sum_{k=0}^p \binom{p}{k} n^k - n^p$$

$$= \sum_{k=0}^{p-1} \binom{p}{k} n^k$$

$$= g(n)$$

$$g(x) = \sum_{k=0}^{p-1} \binom{p}{k} x^k$$

$g(x)$ has degree $p-1$

So by induction

$$\Delta^p \{h_n\}_{n=0}^{\infty} = 0$$

Now

$$\Delta^{p+1} \{n^p\}_{n=0}^{\infty} = \Delta^p \Delta \{n^p\}_{n=0}^{\infty}$$

$$= \Delta^p \{h_n\}_{n=0}^{\infty}$$

$$= 0$$

Result follows.

□

