Math 846

Lecture 28

We are constructing the double cover of the Higman–Sims graph.

Recall the Hoffman–Singleton graph H has 50 vertices. H is strongly-regular with $k = 7$, $a_1 = 0$, $c_2 = 1$.

Each coclique in H has ≤ 15 vertices

$\exists$ coclique C in H with $|C| = 15$.

$\forall x \in H \setminus C$, $|\Gamma(x) \cap H| = 3$

Subgraph induced on $H \setminus C$ is regular with valency 4. It has 35 vertices.

By construction this subgraph is distance-regular with diameter 3 and

c_1	c_2	c_3	k	b_1	b_2
1	1	2.	4	3	3

We recognize, these are the intersection numbers of the Odd graph O_4.

In fact the subgraph is isomorphic to O_4.

Recall the vertex set of O_4 consists of the 3-subsets of a 7-set. Two vertices are adjacent whenever they are disjoint.

Ex. Draw a picture of O_4

By examining how C and $H \setminus C$ are related, we find more cocliques C' in H of size 15.

We find

$\lvert C' \cap C \rvert$	number of cocliques C'
0	7
3	35
5	42
8	15
15	1
	100

H has exactly 100 cocliques of size 15.

Step 3 Higman - Sims graph HS

(100 vertices).

The vertex set of HS consists of the 100

cocliques in H of size 15.

Two vertices in HS are adjacent whenever

their intersection has size 0 or 8.

HS is strongly regular with

$k = 22, \quad c_2 = 6, \quad a_1 = 0$

Moreover

$$\operatorname{Spec}(HS) = \begin{pmatrix} 22 & 2 & -8 \\ 1 & 77 & 22 \end{pmatrix}$$

Aside: the Higman - Sims sporadic finite simple group of size $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 = 44,352,000$ acts on the graph HS as a group of automorphisms.

Step 4 Double cover of HS $2 . HS$

(200 vertices)

Let X = vertex set of HS

Define two copies of X:

$$X^+ = \{ x^+ \mid x \in X \}, \qquad X^- = \{ x^- \mid x \in X \}$$

Vertex set of $2.HS$ is

$$X^+ \cup X^- \qquad \text{(disjoint union)}$$

For $x, y \in X$

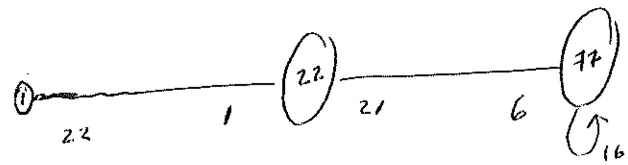
x^+, y^- are adjacent in $2.HS$

whenever

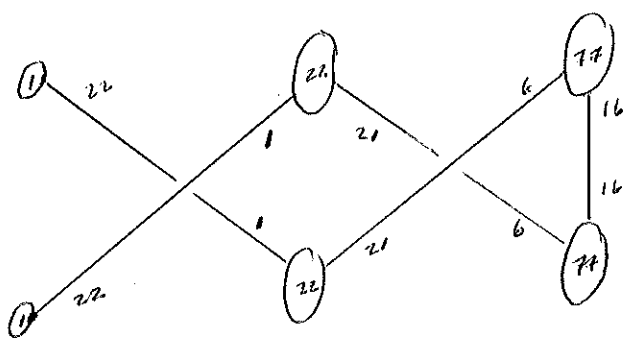
x, y are adjacent in HS

Compare HS and 2.HS

HS:



2.HS



X^-

X^+

2.HS is bipartite with diameter 5 and

c_1	c_2	c_3	c_4	c_5
1	6	16	21	22

So 2.HS is 2-homogeneous with $n=2$

$\square$

We are done discussing the 2-homogeneous DRGs.

CHAPTER 4 The irreducible T-modules

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Until further notice $\Gamma = (X, R)$ is a DRG with diameter $D \geq 1$. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Fix $x \in X$ and write $T = T(x)$ etc.

Recall that the standard module $V = \mathbb{F}^X$ is an orthogonal direct sum of irreducible T-modules.

We now investigate the irred T-modules.

We are mainly interested in the case in which Γ is Q-polynomial.

Let W denote an irred T-module.

Recall:

- The endpt r of W satisfies

$$r = \min\{i \mid 0 \le i \le D,\ E_i^* W \ne 0\}$$

- The dual endpt t of W satisfies

$$t = \min\{i \mid 0 \le i \le D,\ E_i W \ne 0\}$$

- The diameter d of W satisfies

$$d+1 = \left|\{i \mid 0 \le i \le D,\ E_i^* W \ne 0\}\right|$$

- The dual diameter d^* of W satisfies

$$d^*+1 = \left|\{i \mid 0 \le i \le D,\ E_i W \ne 0\}\right|$$

- By construction

$$0 \le r, t, d, d^* \le D$$

$$r+d \le D, \qquad t+d^* \le D$$

LEM 1 For an irred T-module W

(i) $A E_i^* W \subseteq E_{i-1}^* W + E_i^* W + E_{i+1}^* W \qquad (0 \le i \le D)$

(ii) Assume the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial.

Then

$$A^* E_i W \subseteq E_{i-1} W + E_i W + E_{i+1} W \qquad (0 \le i \le D)$$

pf (i) Recall

$$A E_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \qquad (0 \le i \le D)$$

and $E_i^* W = W \cap E_i^* V \qquad (0 \le i \le D)$

(ii) Similar.

$\square$

LEM 2 For an irreducible T-module W with endpt r and diameter d,

$$E_i^* W \neq 0 \quad \text{iff} \quad r \le i \le r+d \qquad (0 \le i \le D)$$

pf By constr $E_i^* W = 0$ for $0 \le i < r$ and

$E_r^* W \neq 0$.

Suppose $\exists\, i$ $(r < i \le r+d)$ st $E_i^* W = 0$.

Define $W' = E_r^* W + E_{r+1}^* W + \cdots + E_{i-1}^* W$

By constr $W' \neq 0$ and

$$A^* W' \subseteq W'$$

Also

$$A W' \subseteq E_r^* W + \cdots + E_{i-1}^* W + \underbrace{E_i^* W}_{0}$$
$$= W'$$

So W' is a T-module.

Now $W' = W$ since the T-module W is irreducible.

This contradicts the diameter of W.

So $\qquad E_i^* W \neq 0 \qquad r \le i \le r+d$

Now. $\qquad E_i^* W = 0 \qquad (r+d < i \le D)$

by the def of diameter d. $\square$

LEM 3 Assume the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial. For an irred T-module W with endpt t and dual diameter d^*

$$E_iW \neq 0 \text{ iff } t \leq i \leq t+d \quad (0 \leq i \leq D)$$

pf Similar to the pf of Lem 2. $\square$