

Combinatorics 475

Lecture 27

Recall the Catalan numbers C_n

thm For $n \geq 0$,

$$C_n = \# \text{ of perms } a_1, a_2, \dots, a_{2n} \text{ of}$$

$$\{1, -1, 1, -1, \dots\}$$

*

such that each partial sum is nonnegative:

$$a_1 + a_2 + \dots + a_i \geq 0 \quad 0 \leq i \leq 2n$$

pf: Assume $n \geq 1$ else trivial

define

$S_n =$ set of all perms of *

$$|S_n| = \binom{2n}{n}$$

define

$A_n =$ set of perms in S_n that have all partial sums nonneg.

show $|A_n| = C_n$

define

$$U_n = S_n \setminus A_n$$

So

$$|A_n| + |U_n| = |S_n|$$

Suf to show

$$|U_n| = \binom{2n}{n-1}$$

Def

 $T_n =$ set of perms of multiset

$$\{(n+1) \cdot 1, (n-1) \cdot 1\}$$

$$|T_n| = \binom{2n}{n-1}$$

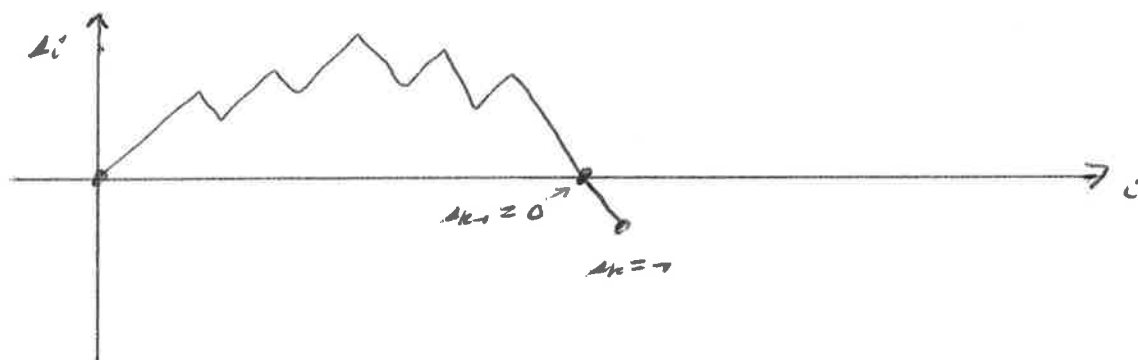
Display bijection

$$f: U_n \rightarrow T_n$$

Given perm
 $a_1 a_2 \dots a_{2n} \in U_n$

This perm has at least one negative partial sum

Pick minimal k such that k th partial sum is neg



k is odd

$$\Delta_k = -1$$

$$\Delta_{k-1} = 0$$

$$a_k = -1$$

$a_1, a_2, \dots, a_{k-1}$ has $\frac{k-1}{2}$ 1's and $\frac{k-1}{2}$ -1's.

Replace each of $a_1, a_2, \dots, a_k$ by its opposite
and leave alone $a_{k+1}, \dots, a_n$:

$$(-a_1)(-a_2) \dots (-a_k) a_{k+1} \dots a_n$$

this sequence has $n-k$ 1's and k -1's

so it is in T_n

this gives function

$$f: U_n \rightarrow T_n$$

f Injective by const

check f surjective:

Given perm

$$b_1, b_2, \dots, b_{2n} \in T_n$$

has $n+1$ 1's
 $n-1$ -1's

Last partial sum is $n+1 - (n-1) = 2 > 0$

Pick minimal k such that k th partial sum is > 0

$$b_1 + b_2 + \dots + b_k = 1$$

$$b_1 + \dots + b_{2n} = 0$$

$$b_k = 1$$

k odd
 $b_1, b_2, \dots, b_{k-1}$ has $\frac{k-1}{2}$ 1's
 $\dots$ $\frac{k-1}{2}$ -1's

For $1 \leq i \leq 2n$ define

$$a_i = \begin{cases} -b_i & \text{if } 1 \leq i \leq k \\ b_i & \text{if } k+1 \leq i \leq 2n \end{cases}$$

then

$$a_1, a_2, \dots, a_{2n} \in U_n$$

and

$$f(a_1, \dots, a_{2n}) = b_1, \dots, b_{2n}$$

So f is surj.

So f is bijectm.

Now

$$|A_n| = \overset{u}{|S_n|} - \overset{u}{|U_n|}$$
$$\qquad \qquad \qquad \overset{u}{\binom{2n}{n}} \qquad \overset{u}{|T_n|}$$
$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \overset{u}{\binom{2n}{n-1}}$$

$$= \binom{2n}{n} - \binom{2n}{n-1}$$

$$= C_n$$

□

A recurrence for C_n

Pf2

Def $A = A_n =$ set of perms $a_1, a_2, \dots, a_n$ of

$\{1, \dots, n\}$ with all partial sums $\Delta_i \geq 0$ for $0 \leq i \leq n$.

Given

$$a_1, a_2, \dots, a_n \in A$$

$$a_1 = 1,$$

&

$$\Delta_0 = 0,$$

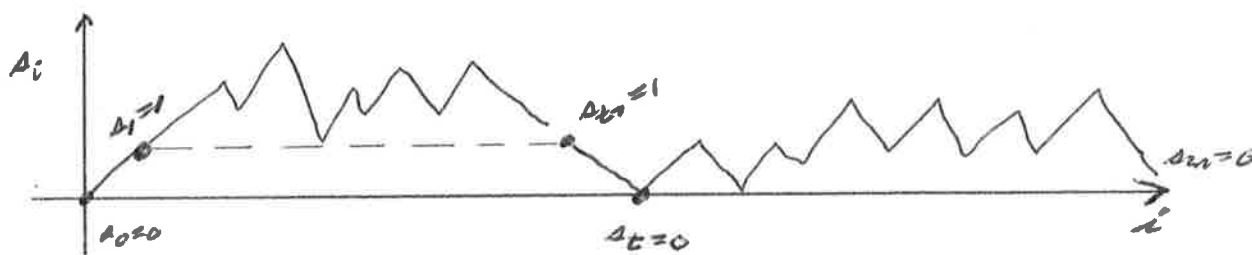
$$\Delta_n = 0$$

$$\Delta_i = 1,$$

Def

$$t = \min \{ i \mid 2 \leq i \leq n, \Delta_i = 0 \}$$

"Landing number"



t even

$$\Delta_1 = a_1 = 1$$

$$\Delta_t = 0$$

$$a_t = -1$$

$$\Delta_{t+1} = 1$$

$$\Delta_i \geq 1 \quad 1 \leq i \leq t-1$$

Partition A according to landing number

For $0 \leq k \leq n-1$ define

$A(k) =$ set of perms in A that have landing number $2k+2$

$\{A(k)\}_{k=0}^{n-1}$ partition A

$$\text{So } |A| = \sum_{k=0}^{n-1} |A(k)|$$

For $0 \leq k \leq n-1$ find $|A(k)|$

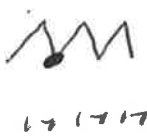
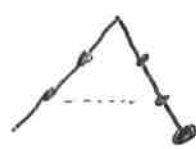
We construct $a_1, a_2, \dots, a_n \in A(k)$ in stages
(write $t = 2k+2$)

stage	to do	# choices
1	pick a_1 ($= 1$)	1
2	pick $a_2, a_3, \dots, a_t$	C_k
3	pick a_t ($= -1$)	1
4	pick $a_{t+1}, \dots, a_n$	C_{n-1-k}

$$\text{So } |A(k)| = C_k C_{n-1-k}$$

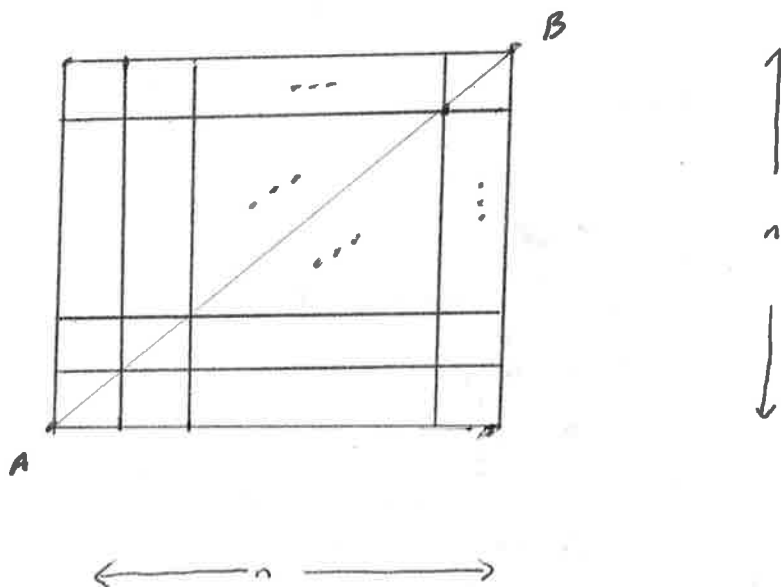
$$\text{So } C_n = |A| = \sum_{k=0}^{n-1} |A(k)| = \sum_{k=0}^{n-1} C_k C_{n-1-k} \quad \square$$

ex $n=3$

k	$A(k)$		$ A(k) $
0			$2 = c_0 c_2$
1			$1 = c_1 c_1$
2			$2 = c_2 c_0$
			$5 = c_3$

Ex For $n \geq 1$,

consider city blocks



How many paths of length $2n$ from A to B
that do not cross the diagonal?



Let $h_n = \#$ paths from A to B that lie to NW of diagonal

Obs $h_n = \dots$ SE

Ans = $2h_n$

From h_n

Each path from A to B is sequence

$$a_1, a_2, \dots, a_{2n}$$

$$a_i \in \{N, E\} \quad 1 \leq i \leq 2n$$

$$\text{View } N \leftrightarrow 1$$

$$E \leftrightarrow -1$$

path stays NW of diag $\Leftrightarrow$ all partial sums ≥ 0

$$h_n = C_n$$

$$\text{Ans} = 2C_n$$

$$= \frac{2}{n+1} \binom{2n}{n}$$

□

