Math 846

Lecture 27

We continue to discuss the 2-homogeneous graphs with $D = 3, 4, 5$

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Case $D = 4$ Assume $\Gamma = (X, R)$ is

2-homog with valency $k \geq 2$

By thm 43 case (iv), $|X| = 4k$ and

θ_0^*	θ_1^*	θ_2^*	θ_3^*	θ_4^*
k	$\sqrt{k}$	0	$-\sqrt{k}$	$-k$

Write $E = E_1$.

$\forall x, y \in X$,

$$\langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta_i^* \qquad i = \partial(x,y)$$

So

$\partial(x,y) = 4$ iff $E\hat{x} = -E\hat{y}$

$\partial(x,y) = 2$ iff $\langle E\hat{x}, E\hat{y} \rangle = 0$

$\partial(x,y) = 0$ iff $E\hat{x} = E\hat{y}$

Bipartition $X = X^+ \cup X^-$ (disjoint union)

$$|X^+| = |X^-| = 2k$$

Define $\Phi^+ = \{ E\hat{x} \mid x \in X^+ \}$

$$|\Phi^+| = 2k$$

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$\forall u, v \in \Phi^+$

$u = v$ or $u = -v$ or $\langle u, v \rangle = 0$

Also $u \in \Phi^+$ implies $-u \in \Phi^+$

So $\Phi^+ = \{ \pm u_1, \pm u_2, \dots, \pm u_k \}$

$\| u_i \|^2 = |X|^{-1} k \qquad 1 \le i \le k$

$\langle u_i, u_j \rangle = 0 \text{ if } i \ne j \qquad 1 \le i, j \le k$

Def $\Phi^- = \{ E\hat{x} \mid x \in X^- \}$

Similarly $\Phi^- = \{ \pm v_1, \pm v_2, \dots, \pm v_k \}$

$\| v_i \|^2 = |X|^{-1} k \qquad 1 \le i \le k$

$\langle v_i, v_j \rangle = 0 \text{ if } i \ne j \qquad 1 \le i, j \le k$

- $\{ u_i \}_{i=1}^{k}$ is orthog basis for EV
- $\{ v_i \}_{i=1}^{k}$ ...

Define $k \times k$ matrix H:

For $1 \le i, j \le k$

$$H_{ij} = \frac{|X| \langle u_i, v_j \rangle}{\sqrt{k}}$$

Note
$$\langle u_i, v_j \rangle = |X|^{-1} \theta_\ell^* \quad \ell = 1 \text{ or } 3$$
$$= \pm |X|^{-1} \sqrt{k}$$

(*)

So $H_{ij} \in \{1, -1\}$

- The transition matrix from $\{u_i\}_{i=1}^k$ to $\{v_i\}_{i=1}^k$ is $\frac{H}{\sqrt{k}}$

pf Let S = trans matrix

For $1 \le j \le k$
$$v_j = \sum_{\ell=1}^k S_{\ell j} u_\ell$$

so for $1 \le i \le k$
$$\langle u_i, v_j \rangle = \langle u_i, \sum_{\ell=1}^k S_{\ell j} u_\ell \rangle$$

$$\sqrt{k} H_{ij} |X|^{-1} = \langle u_i, v_j \rangle = S_{ij} \| u_i \|^2 = S_{ij} |X|^{-1} k$$

so $H_{ij} = S_{ij} \sqrt{k}$ ✓

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- The transition matrix from $\{v_i\}_{i=1}^{k}$ to $\{u_i\}_{i=1}^{k}$

is $\frac{H^t}{\sqrt{k}}$.

- By linear algebra,

$$\frac{H}{\sqrt{k}} \frac{H^t}{\sqrt{k}} = I = \frac{H^t}{\sqrt{k}} \frac{H}{\sqrt{k}}$$

So

$$HH^t = kI = H^t H \qquad (**)$$

Any $k \times k$ matrix H satisfying $(*)$, $(**)$

is called <u>Hadamard</u>

Next we reverse the logical direction.

Given a Hadamard matrix H

we define a graph $\Gamma = (X, R)$ as follows.

Let H be $k \times k$ ($k \geq 2$). Define

$$X = X^+ \cup X^- \quad \text{(disjoint union)}$$

where

$$X^+ = \{\pm r_1, \pm r_2, \dots, \pm r_k\}$$

$$X^- = \{\pm c_1, \pm c_2, \dots, \pm c_k\}$$

For $1 \leq i, j \leq k$

if $H_{ij} = 1$ then in Γ

if $H_{ij} = -1$ then in Γ

then Γ is a 2-homogeneous DRG with diam $D = 4$ and valency k. (ex)

Case $D = 5$ Assume Γ is 2-homogeneous

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Recall the intersection numbers of Γ:

c_1	c_2	c_3	c_4	c_5
1	n^2+n	$n^2(n+2)$	$(n+1)(n^2+2n-1)$	$n(n^2+3n+1)$

$n \in \mathbb{Z}$ $n \geq 2$

For $n=2$, Γ exists and is unique up to isomorphism;
it is called the double cover of the Higman-Sims graph.

For $n=3$ Γ does not exist.

For $n \geq 4$ the existence Γ is unknown

<u>Open Problem</u> For $n \geq 4$ determine if Γ
exists or not.

Next goal: Describe the double cover of the Higman-Sims graph

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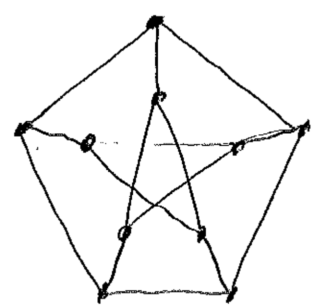
Step 1 Recall Peterson graph P

P :



Cyclic group $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$

$X = \{ x_i, y_i \mid i \in \mathbb{Z}_5 \}$

For $i, j \in \mathbb{Z}_5$

x_i, y_j adj iff $i = j$

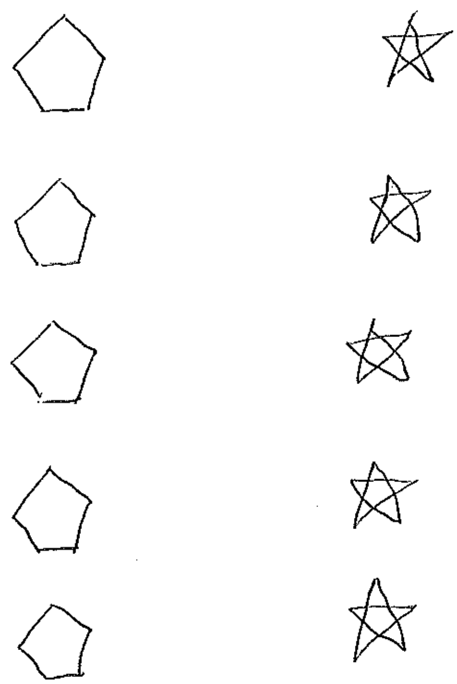
x_i, x_j adj iff $i - j \in \{1, -1\}$

y_i, y_j adj iff $i - j \in \{2, -2\}$

Step 2 Hoffman-Singleton graph H

(50 vertices)

Consider 5 pentagons and 5 pentagrams:



Connect each  to each  using 5 edges

in order to create a Peterson graph.

Create these connections so that H has no

4-cycles.

To do this, for $h, i, j \in \mathbb{Z}_5$ connect vertex j

of pentagon h to vertex $hi + j$ of pentagram i

The graph H is regular with valency $k=7$

H has no 3-cycle or 4-cycle.

Each vertex of H is at distance ≤ 2 from

$$1+7+7\cdot 6=50$$

vertices.

So H is connected with diameter 2

Moreover H is strongly-regular

with intersection numbers

$k=7$, $c_2=1$, $a_1=0$

Find spectrum of H:

$v_0=1$, $v_1=\lambda$, $v_2=\lambda^2-7$,

$$v_3=\lambda^3-6\lambda^2-13\lambda+42$$
$$=(\lambda-7)(\lambda-2)(\lambda+3)$$

$\theta_0=7$, $\theta_1=2$, $\theta_2=-3$

	$v_0 = 1$	$v_1 = \lambda$	$v_2 = \lambda^2 - 7$
$\theta_0 = 7$	1	7	42
$\theta_1 = 2$	1	2	-3
$\theta_2 = -3$	1	-3	2

	$u_0 = 1$	$u_1 = \frac{\lambda}{7}$	$u_2 = \frac{\lambda^2 - 7}{42}$
$\theta_0 = 7$	1	1	1
$\theta_1 = 2$	1	$\frac{2}{7}$	$\frac{-3}{42} = -\frac{1}{14}$
$\theta_2 = -3$	1	$\frac{-3}{7}$	$\frac{2}{42} = \frac{1}{21}$

For H

$$m_1 = \frac{50}{1 + 2 \cdot \frac{2}{7} + (-3)(-\frac{1}{14})} = 28$$

$$m_2 = \frac{50}{1 + (-3)(\frac{-3}{7}) + 2 \frac{1}{21}} = 21$$

$$\mathrm{Spec}(H) = \begin{pmatrix} 7 & 2 & -3 \\ 1 & 28 & 21 \end{pmatrix}$$

A <u>coclique</u> in any graph is a set of distinct vertices, no two adjacent.

Find the maximal size N of a coclique in H

Given coclique C in H with $|C| = N$

Consider
$$\{ E_1 \hat{x} \mid x \in C \}$$

Matrix of inner products is $|X|^{-1} m_1$ times

$$\begin{pmatrix} 1 & & & -\frac{1}{14} \\ & 1 & & \\ & & \ddots & \\ -\frac{1}{14} & & & 1 \end{pmatrix} \qquad *$$

$*$ is pos semi def.

Spectrum of $*$ is

$$\begin{cases} 1 - \frac{1}{14}(N-1) & \text{mult } 1 \\ 1 - \frac{1}{14}(-1) & \text{mult } N-1 \end{cases}$$

Require

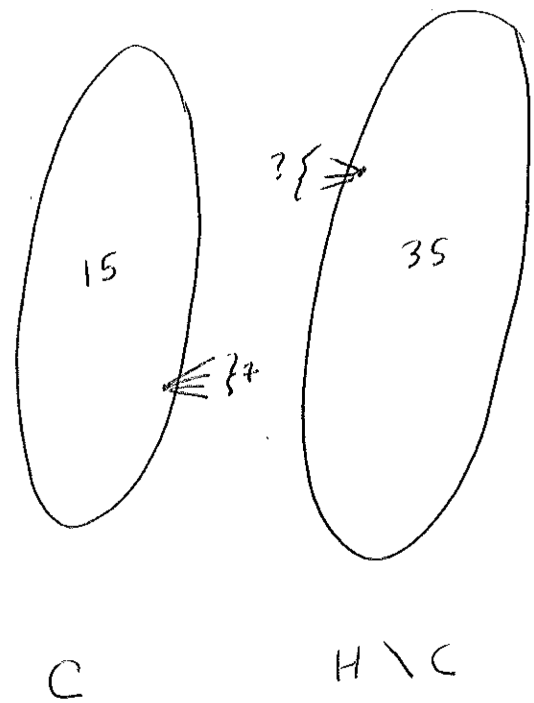
$$1 - \frac{N-1}{14} \geq 0$$

So $\qquad N \leq 15$

By displaying a coclique of size 15, we find

$N = 15$

H:



For a vertex y of H not in C

find # vertices in C that are adj y.

obs

$$\Big\| \sum_{x \in C} E_1 \hat{x} \Big\|^2 = N |X|^{-1} m_1 \left(1 - \frac{N-1}{14}\right)$$
$$= 0$$

So

$$\sum_{x \in C} E_1 \hat{x} = 0$$

So

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$$0 = \left\langle \sum_{x \in C} E_1 \hat{x},\ E_1 \hat{y} \right\rangle$$

$$= |X|^{-1} m_1 \left(\# \frac{2}{7} + (15 - \#) \frac{-1}{14} \right)$$

So $\# = 3$

Every vertex in $H \setminus C$ is adjacent exactly 3 vertices in C

(more steps to come)