

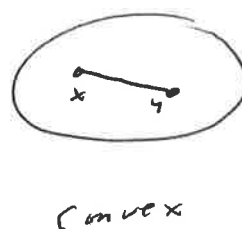
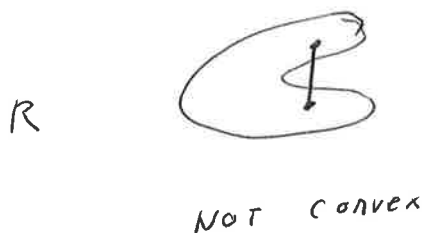
Combinatorics 475

Lecture 26

7.6 A geometry example

Given a subset R of the plane

R is convex whenever for all $x, y \in R$ the line segment $\overline{xy}$ is contained in R

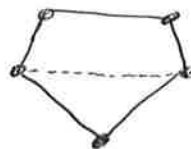
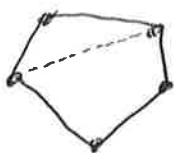


Consider convex polygon

ex 5-gon



A diagonal is a line segment joining nonadjacent corners



For convex n -gon

diagonals is $\binom{n}{2} - n = \frac{n(n-3)}{2}$

Problem For $n \geq 2$ let

$h_n =$ # ways to divide a convex $(n+1)$ -gon into triangular regions, by inserting diagonals that do not intersect

For notational convenience def $h_1 = 1$

Find h_n

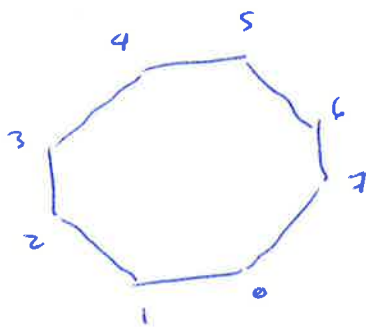
Ex

n	desc	h_n
1		1
2		1
3		2
4		5
:		:

To find h_n , find a recurrence
it satisfies

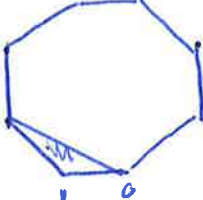
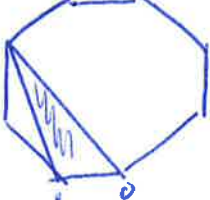
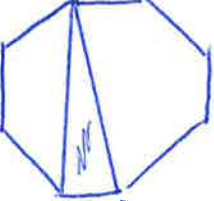
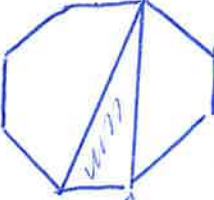
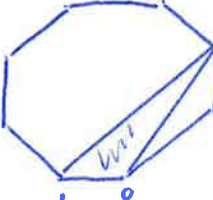
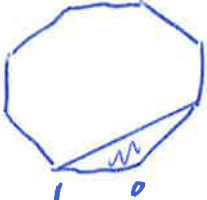
ex $n=7$

Label the vertices of convex 8-gon $0, 1, \dots, 7$



Consider a triangular division of this 8-gon
edge 01 is contained in a triangle $01t$ some t
($2 \leq t \leq 7$)

Cases

t	desc	# choices
2		$h_6 = h_1 h_6$
3		$h_5 = h_2 h_5$
4		$h_3 h_4$
5		$h_4 h_3$
6		$h_5 h_2$
7		$h_6 h_1$

$$h_7 = h_1 h_6 + h_2 h_5 + \dots + h_6 h_1$$

$$h_7 = \sum_{k=1}^6 h_k h_{7-k}$$

In gen for $n \geq 2$

$$h_n = \sum_{k=1}^{n-1} h_k h_{n-k}$$

To solve this rec use gen function

$$g(x) = \sum_{n=1}^{\infty} h_n x^n$$

Note $g(x)$ has no constant term $\Rightarrow$

$$g(0) = 0$$

$$\text{Obs } (g(x))^2 = \left(\sum_{r=1}^{\infty} h_r x^r \right) \left(\sum_{s=1}^{\infty} h_s x^s \right)$$

$$= \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} h_r h_s x^{r+s}$$

$$= \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} h_k h_{n-k} \right) x^n$$

$$= \sum_{n=2}^{\infty} h_n x^n$$

$$= g(x) - h_1 x$$

$$h_1 = 1$$

$$= g(x) - x$$

$$(g(x))^2 - g(x) + x = 0$$

Solve for $g(x)$ using quadratic formula

$$g(x) = \frac{1 + \sqrt{1-4x}}{2} \quad \text{or} \quad g(x) = \frac{1 - \sqrt{1-4x}}{2}$$

Which sign is correct?

take $x=0$

$$g(0) = 0$$

$$0 \stackrel{?}{=} \frac{1 + \sqrt{1-0}}{2}$$

NO

$$0 = \frac{1 - \sqrt{1-0}}{2}$$

YES

$$\begin{aligned} g(x) &= \frac{1 - \sqrt{1-4x}}{2} \\ &= \frac{1 - (1-4x)^{1/2}}{2} \end{aligned}$$

Apply Newton binom thm

$$= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

For $n \geq 1$

$$h_n = -\frac{1}{2} \binom{\frac{1}{2}}{n} (-4)^n$$

$$= \frac{1}{n} \binom{2n-2}{n-1}$$

ex $n=6$

$$-\frac{1}{2} \binom{\frac{1}{2}}{6} (-4)^6 = -\frac{1}{2} \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2} \cdot -\frac{7}{2} \cdot -\frac{9}{2}}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = -4 - 4 - 4 - 4 - 4 - 4$$

ch 8 Special Counting Sequences

8.1 Catalan numbers

For $n \geq 0$ define

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

"nth Catalan number"

n	0	1	2	3	4	5	...
C_n	1	1	2	5	14	42	...

In Section 7.6 we considered triangular divisions
of a convex $(n+1)$ -gon

the answer was

$$\frac{1}{n} \binom{2n-2}{n-1} = C_{n-1}$$

LEM F_n $n \geq 1$

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}$$

pf

$$\frac{1}{n+1} \binom{2n}{n} \stackrel{?}{=} \binom{2n}{n} - \binom{2n}{n+1}$$

$$\begin{aligned} \binom{2n}{n+1} &\stackrel{?}{=} \binom{2n}{n} - \frac{1}{n+1} \binom{2n}{n} \\ &= \frac{n}{n+1} \binom{2n}{n} \end{aligned}$$

$$\frac{(2n)!}{(n+1)!(n)!} \stackrel{?}{=} \frac{n}{n+1} \frac{(2n)!}{n!n!}$$

✓

□

An interpretation of C_n

Fix integer $n \geq 0$

Consider a permutation

$$a_1, a_2, \dots, a_{2n}$$

of multiset

$$\{ n \cdot 1, n \cdot -1 \}$$

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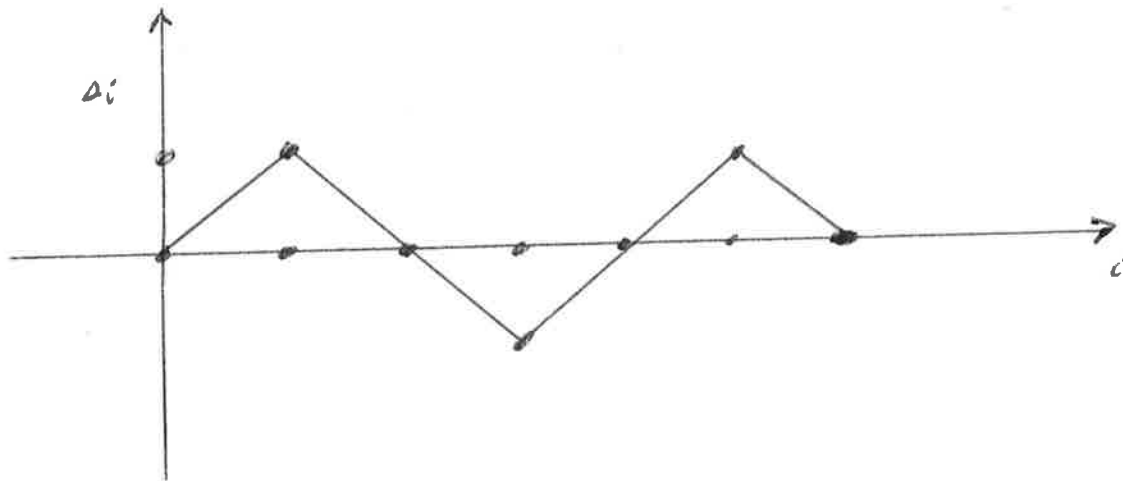
For $0 \leq i \leq 2n$ the i th partial sum is

$$A_i = a_1 + a_2 + \dots + a_i$$

ex $n = 3$

$$a_1, a_2, a_3, a_4, a_5, a_6 = 1, -1, -1, 1, 1, -1$$

i	0	1	2	3	4	5	6
a_i	0	1	0	-1	0	1	0

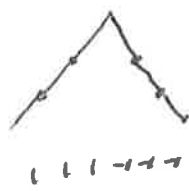


Problem

Let

$h_n = \# \text{ perms of } * \text{ that have}$
all partial sums nonnegative.

Find h_n .

n	desc	h_n
0		1
1		1
2	 	2
3	  	5
	 