Math 846

Lecture 26

We continue to discuss a DRG $\Gamma = (X, R)$ with diameter $D \geq 1$ that is bipartite and has a Q-poly ordering $\{E_i\}_{i=0}^{D}$ that is dual bipartite.

Def 40 For $0 \neq q \in \mathbb{C}$ the algebra $U_q(so_3)$ is defined by generators a, b, c and rels

$$
\begin{aligned}
qab - q^{-1}ba &= c,\\
qbc - q^{-1}cb &= a,\\
qca - q^{-1}ac &= b.
\end{aligned}
$$

Fix $x \in X$ and write $T = T(x)$

With ref to Lem 25 assume $\beta \neq \mp 2$

and write $\beta = q^2 + q^{-2}$

By LEM 27,

$$q AB - q^{-1} BA = z C,$$
$$q BC - q^{-1} CB = z A,$$
$$q CA - q^{-1} AC = z B$$

where

$$z = i^{0} (q^{0-2} + q^{2-0}) \qquad i^{0\,2} = -1$$

So $\exists$ surjective algebra hom

$$U_q(so_3) \to T$$

that sends

$$a \to \frac{A}{z}$$

$$b \to \frac{B}{z}$$

$$c \to \frac{C}{z}$$

Combinatorial aspects.

LEM 41 Assume that $\Gamma = (X, R)$ is a bipartite DRG with diam $D \geq 2$ and $c_2 \geq 2$ (not nec Q-poly). Then

$$p^i_{2i} \neq 0 \qquad (1 \leq i \leq D-1)$$

pf Pick $x, y \in X$ at $\partial(x,y) = i$.

Show $\Gamma_i(x) \cap \Gamma_2(y) \neq \emptyset$.

$\exists \ w \in \Gamma_{i-1}(x) \cap \Gamma(y)$

$\exists \ w' \in \Gamma_{i+1}(x) \cap \Gamma(y)$

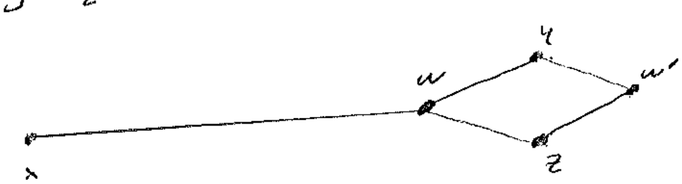
We have $\partial(w, w') = 2$

$|\Gamma(w) \cap \Gamma(w')| = c_2 \geq 2$

$y \in \Gamma(w) \cap \Gamma(w')$

$\exists \ z \in \Gamma(w) \cap \Gamma(w') \setminus y$

We have



(☆)

Observe $z \in \Gamma_i(x) \cap \Gamma_2(y)$

So $p^i_{2i} \neq 0$ □

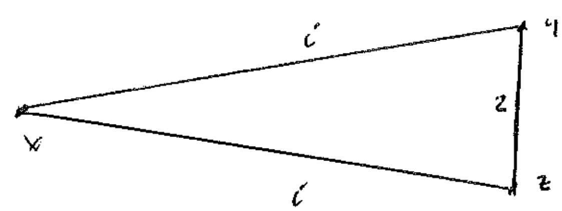
Assume our DRG $\Gamma = (X, R)$ is bipartite with diameter $D \geq 2$, and the Q-poly ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite.

Assume Γ is not a $2D$-cycle, so $c_2 \geq 2$ by the note below Lem 25. Now by LEM 41

$$p^i_{2i} \neq 0 \qquad (1 \leq i \leq D-1)$$

For $1 \leq i \leq D-1$, $\exists$ $x, y, z \in X$ st

$$\partial(x,y) = i, \qquad \partial(x,z) = i, \qquad \partial(y,z) = 2$$

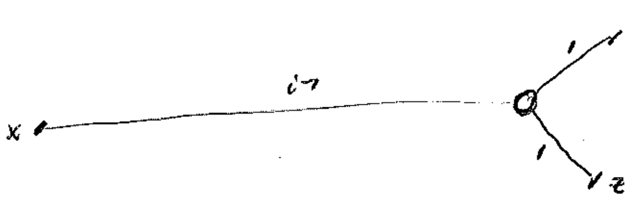


We have $$|\Gamma(y) \cap \Gamma(z)| = c_2$$

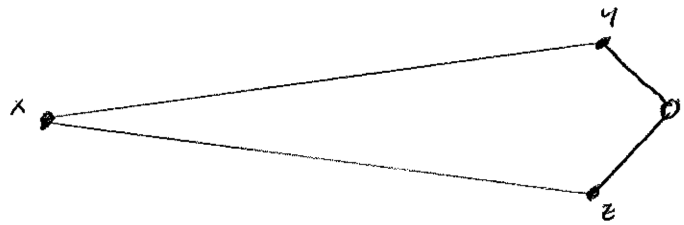
Partition $$\Gamma(y) \cap \Gamma(z)$$

into
$$\Gamma_{i-1}(x) \cap \Gamma(y) \cap \Gamma(z) \qquad (*)$$
$$\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z) \qquad (**)$$

$*$:



$**$:



Next goal: find

$$|*|, \quad |**|$$

By constr $|*| + |**| = c_2$

For the hypercube $H(D,2)$, $c_2 = 2$ and

$$|*| = 1, \quad |**| = 1$$ by Lem 17, CH1

Next assume Γ is not a hypercube.

Then $D \geq 3$ and $\beta \neq \mp 2$ (β from THM 1)

Write $\beta = q^2 + q^{-2}$ as in LEM 25

Prop 42. With the above notation

for $1 \le i \le D-1$ Pick x, y, z s.t.

$$\partial(x,y) = i, \quad \partial(x,z) = i, \quad \partial(y,z) = 2$$

Then

$$|\Gamma_{i-1}(x) \cap \Gamma(y) \cap \Gamma(z)| = \frac{q^{D-2i-2} + q^{2i+2-D}}{q^{D-2i} + q^{2i-D}} \; \frac{q^{D-2} + q^{2-D}}{q^{D-4} + q^{4-D}} \qquad (1)$$

$$\#_i \; /\!/$$

$$|\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = \frac{q^{D-2i+2} + q^{2i-2-D}}{q^{D-2i} + q^{2i-D}} \; \frac{q^{D-2} + q^{2-D}}{q^{D-4} + q^{4-D}} \qquad (2)$$

pf Write $A^* = A^*(x)$ and recall

$$A^2A^* - (q^2+q^{-2}) AA^*A + A^*A^2 = (q^{D-2} + q^{2-D})^2 A^*$$

Compute (y,z)-entry of each side:

$$c_2 \theta_i^* - (q^2+q^{-2}) \left(\#_i \theta_{i-1}^* + (c_2 - \#_i)\theta_{i+1}^* \right) + c_2 \theta_i^* = 0$$

Solve for $\#_i$ to get

$$\#_i = \frac{c_2}{q^2+q^{-2}} \; \frac{2\theta_i^* - (q^2+q^{-2})\theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*}$$

Evaluate $\#_i$ using the data in Lem 25

to get (1)

To get (2), observe

$$\text{RHS of (1)} + \text{RHS of (2)} = c_2$$

$\square$

Ref to Prop 42 for $1 \le i \le D-1$

$$\#_i \text{ is an integer}$$

By (☆),

$$1 \le \#_i \le c_2 - 1$$

By constr $\#_1 = 1$

Abbrev $\#_2 = r$

Next goal: make a change of variables and express everything in terms of r instead of q

Thm 43 (Kazumasa Nomura) Assume the DRG $\Gamma = (X, R)$ is bipartite with diameter $D \geq 1$. Then Γ has a dual bipartite ordering $\{E_i\}_{i=0}^{D}$ if and only if either:

(i) Γ is a $2D$-cycle

(ii) Γ is a hypercube $H(D, 2)$

(iii) $D = 3$ and

c_1	c_2	c_3	k	b_1	b_2
1	$n+1$	$n+2$	$n+2$	$n+1$	1

$n \geq 2,\ n \in \mathbb{Z}$

(iv) $D = 4$ and

c_1	c_2	c_3	c_4	k	b_1	b_2	b_3
1	$2n$	$4n-1$	$4n$	$4n$	$4n-1$	$2n$	1

$n \geq 2,\ n \in \mathbb{Z}$

(v) $D = 5$ and

c_1	c_2	c_3	c_4	c_5
1	n^2+n	$n^2(n+2)$	$(n+1)(n^2+2n-1)$	$n(n^2+3n+1)$

k	b_1	b_2	b_3	b_4
$n(n^2+3n+1)$	$(n+1)(n^2+2n-1)$	$n^2(n+2)$	n^2+n	1

$n \geq 2,\ n \in \mathbb{Z}$

Assume the equiv conditions hold.

For $0 \le i \le D$ let θ_i = eigenvalue of Γ for E_i

For Case (iii)

#1	#2
1	n

θ_0	θ_1	θ_2	θ_3
$n+2$	1	-1	$-n-2$

For Case (iv)

#1	#2	#3
1	n	$2n-1$

θ_0	θ_1	θ_2	θ_3	θ_4
$4n$	$2\sqrt{n}$	0	$-2\sqrt{n}$	$-4n$

For Case (v)

#1	#2	#3	#4
1	n	n^2	n^2+n-1

θ_0	θ_1	θ_2	θ_3	θ_4	θ_5
$n(n^2+3n+1)$	n^2+2n	n	$-n$	$-n^2-2n$	$-n(n^2+3n+1)$

In all cases

$$\theta_i^* = \theta_i, \qquad c_i^* = c_i, \qquad b_i^* = b_i \qquad (0 \le i \le D).$$

pf. Assume Γ has a dual bipartite Q-poly ordering $\{E_i\}_{i=0}^D$.

Assume Γ is not a 2D-cycle or $H(D,2)$

We show Γ satisfies (iii) or (iv) or (v)

We have $D \geq 3$ and $c_2 \geq 2$ and

$\beta \neq \pm 2$ (β from Thm 1)

Write $\beta = q^2 + q^{-2}$ as in Lem 25

By Prop 42

$$n - 1 = \left(\frac{q^2 - q^{-2}}{q^{D-4} + q^{4-D}} \right)^2$$

$$= \frac{(\beta - 2)(\beta + 2)}{(q^{D-4} + q^{4-D})^2}$$

$$\neq 0$$

so $n \geq 2$

In each case $D = 3, 4, 5$ we routinely verify the tables using Lem 25

Show $D \geq 6$ is impossible.

Suppose $D \geq 6$

Then $b_5 \geq 1$

So $b_5 - 1 \geq 0$

Using the data in Lem 25 and Prop 42

$$b_5 - 1 = - \frac{\overset{+}{n}\,\overset{+}{(c_2-1)}\,\overset{+}{(nc_2 - 2c_2 + 2n)}}{\underset{+}{(c_2^2 n - c_2^2 + n^2)}\,\underset{+}{(n-1)}}$$

< 0

Cont.

So $D \leq 6$.

We have shown the Thm in one direction

The converse is routinely checked using

Th 67 in CH2.

The remaining tables are checked using Lem 25

and Prop 42

$\square$

DEF 44 The graphs Γ in Thm 43 are called <u>2-homogeneous</u>

Next we describe in more detail the 2-homogeneous graphs with diameter 3, 4, 5.

<u>Case $D=3$</u> For $k \geq 2$, up to isomorphism $\exists$ unique 2-homogeneous graph $\Gamma = (X, \mathcal{R})$ with diameter $D=3$ and valency k. Γ is described as follows:

$$X = \{\pm 1, \pm 2, \dots, \pm(k+1)\}$$

vertices $x, y \in X$ are adjacent whenever

$$x+y \neq 0$$

and x, y have opposite sign.