

Combinatorics 475

Lecture 25

7.5

Non homogeneous

Recurrence Relations

Lec 25

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Given a sequence $\{h_n\}_{n=0}^{\infty}$ of real numbers

Suppose it satisfies a rec such as

$$h_n = a h_{n-1} + b h_{n-2} + c$$

$$n \geq 2$$

where

a, b indep of n

$c =$ dep on n

Sol depends on nature of function c

We illustrate with examples

Ex Fix $a, c \in \mathbb{R}$ Assume

$$h_n = a h_{n-1} + c \quad n \geq 1$$

Find h_n

Sol (Examine small n)

n	h_n
0	h_0
1	$a h_0 + c$
2	$a(a h_0 + c) + c = a^2 h_0 + c(1+a)$
3	$a(a^2 h_0 + (1+a)c) + c$ $= a^3 h_0 + (1+a+a^2)c$
$\vdots$	$\vdots$
n	$a^n h_0 + (1+a+a^2+\dots+a^{n-1})c$ $= a^n h_0 + \frac{1-a^n}{1-a} c$

Ex Solve the recurrence

$$h_n = 4h_{n-1} - 4h_{n-2} + 2 \quad n \geq 2$$

$$h_0 = 3, \quad h_1 = 6$$

Sol (reduce to hom case)

$$h_n = 4h_{n-1} - 4h_{n-2} + 2$$

$$n \rightarrow n+1 \quad h_{n+1} = 4h_n - 4h_{n-1} + 2$$

subtract

$$h_n = 5h_{n-1} - 8h_{n-2} + 4h_{n-3} \quad n \geq 3$$

Now solve using prev methods

Factor char poly

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2(x-1)$$

Gen sol is

$$h_n = (a+bt)2^n + c1^n$$

$$n = 0, 1, 2, \dots$$

Find c

$$Z = h_n - 4h_{n-1} + 4h_{n-2}$$

$$= c - 4c + 4c$$

$$= c$$

$$h_n = (a + b)2^n + 2$$

Use init cond to find a, b

$$a = 1$$

$$b = 1$$

$$h_n = (n+1)2^n + 2$$

$$n = 0, 1, 2, \dots$$

□

Ex Solve the recurrence

$$h_n = 6h_{n-1} - 9h_{n-2} + 2n \quad n \geq 2$$

*

$$h_0 = 1, \quad h_1 = 0$$

Sol (Red to hom case)

Note the sequence

$$a_n = n$$

satisfies

$$a_n - 2a_{n-1} + a_{n-2} = 0 \quad n \geq 2$$

Use this to get rid of "2n" term in *

Write

$$h_n - 6h_{n-1} + 9h_{n-2} = 2n$$

coef	h_n	h_{n-1}	h_{n-2}	h_{n-3}	h_{n-4}	const
1	1	-6	9			$2n$
-2		1	-6	9		$2(n-1)$
1			1	-6	9	$2(n-2)$
	1	-8	22	-24	9	0



$$h_n - 8h_{n-1} + 22h_{n-2} - 24h_{n-3} + 9h_{n-4} = 0 \quad n \geq 4$$

Factor charpoly

$$x^4 - 8x^3 + 22x^2 - 24x + 9 = (x-3)^2(x-1)^2$$

Gen sol to  is

$$h_n = (an+b)3^n + cn+d$$

$n=0, 1, 2, \dots$

Find c, d

$$2n = h_n - 6h_{n-1} + 9h_{n-2}$$

$$= cn+d - 6(cn+1+d) + 9(c(n-2)+d)$$

$$\begin{aligned} n \text{ coef: } 2 &= c - 6c + 9c \\ &= 4c \end{aligned}$$

$$c = 1/2$$

$$\rightarrow d = 3/2$$

So far

$$h_n = (an + b) 3^n + \frac{n+3}{2}$$

Find a, b

Use init cond $h_0 = 1, h_1 = 0$

to get

$$a = -1/6, \quad b = -1/2$$

Now

$$h_n = -\frac{n+3}{6} 3^n + \frac{n+3}{2}$$

$$= \frac{(3 - 3^n)(n+3)}{6}$$

$$n = 0, 1, 2, \dots$$

□

7.5 cont

Ex Solve the rec

$$h_n = 2h_{n-1} + 5^n \quad n \geq 1$$

$$h_0 = 3$$

Sol Note $h_1 = 11$

$$h_n - 2h_{n-1} = 5^n$$

$$h_{n-1} - 2h_{n-2} = 5^{n-1}$$

$$0 = h_n - 2h_{n-1} - 5(h_{n-1} - 2h_{n-2})$$

$$= h_n - 7h_{n-1} + 10h_{n-2} \quad n \geq 2$$

char poly

$$x^2 - 7x + 10 = (x-5)(x-2)$$

General

$$h_n = a5^n + b2^n$$

$$\text{Using } h_0 = 3, \quad h_1 = 11$$

$$a = 5/3 \quad b = 4/3$$

$$h_n = \frac{5^{n+1} + 2^{n+2}}{3}$$

$$n = 0, 1, 2, \dots$$

□

Ex Solve the rec

$$h_n = 5h_{n-1} + 5^n \quad n \geq 1$$

$$h_0 = 3$$

Sol Note

$$\begin{aligned} h_1 &= 5 \cdot 3 + 5 \\ &= 20 \end{aligned}$$

$$h_n - 5h_{n-1} = 5^n \quad (1)$$

$$h_{n-1} - 5h_{n-2} = 5^{n-1} \quad - (5)$$

$$\begin{aligned} 0 &= h_n - 5h_{n-1} - 5(h_{n-1} - 5h_{n-2}) \\ &= h_n - 10h_{n-1} + 25h_{n-2} \end{aligned}$$

char poly

$$x^2 - 10x + 25 = (x-5)^2$$

Gen sol

$$h_n = (a + b)5^n$$

$$n = 0, 1, 2, \dots$$

Using $h_0 = 3, \quad h_1 = 20$ get

$$a = 1$$

$$b = 3$$

$$h_n = (n+3)5^n$$

$$n = 0, 1, 2, \dots$$

□

ex Solve the recurrence

$$h_n = 3h_{n-1} + n \quad n \geq 1$$

$$h_0 = 2$$

Using gen functions

Sol Define

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

Obs

$$\sum_{n=1}^{\infty} h_{n-1} x^n = \sum_{n=0}^{\infty} h_n x^{n+1} = x g(x)$$

Recall

$$\begin{aligned} \sum_{n=0}^{\infty} n x^n &= \sum_{n=1}^{\infty} n x^n \\ &= x \sum_{n=1}^{\infty} n x^{n-1} \\ &= x \sum_{n=0}^{\infty} (n+1) x^n \\ &= \frac{x}{(1-x)^2} \end{aligned}$$

Obs

$$g(x) - 3xg(x) - \frac{x}{(1-x)^2} = h_0 = 2$$

$$g(x)(1-3x) = \frac{x}{(1-x)^2} + 2$$

$$g(x) = \frac{x}{(1-x)^2(1-3x)} + \frac{2}{1-3x}$$

↑ red using partial frac

$$= \frac{11}{4(1-3x)} + \frac{x-3}{4(1-x)^2}$$

$$= \frac{11}{4} \sum_{n=0}^{\infty} 3^n x^n + \frac{1}{4} \sum_{n=0}^{\infty} n x^n - \frac{3}{4} \sum_{n=0}^{\infty} (n+1) x^n$$

$$= \sum_{n=0}^{\infty} \frac{11 \cdot 3^n - 2n - 3}{4} x^n$$

$$h_n = \frac{11 \cdot 3^n - 2n - 3}{4} \quad n = 0, 1, 2, \dots$$

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