Math 846

Lecture 25

Prop 34 With refrence to Def 28,

(i) $$W = f \frac{\sum_{i=0}^{p} \tau_i^{-1} A_i}{\sum_{i=0}^{p} \tau_i^{-1} k_i}$$

(ii) $$W^{-1} = \frac{1}{f} \frac{\sum_{i=0}^{p} \tau_i A_i}{\sum_{i=0}^{p} \tau_i k_i}$$

(iii) $$W^* = f \frac{\sum_{i=0}^{p} \tau_i^{-1} A_i^*}{\sum_{i=0}^{p} \tau_i^{-1} k_i}$$

(iv) $$(W^*)^{-1} = \frac{1}{f} \frac{\sum_{i=0}^{p} \tau_i A_i^*}{\sum_{i=0}^{p} \tau_i k_i}$$

pf (i) We have

$$W^{-1} E_0^* W = W^* E_0 (W^*)^{-1}$$

so

$$\underbrace{E_0^* W^{-1} E_0^*}_{\displaystyle \frac{\sum_{i=0}^{D} \tau_i^{-1} k_i}{f|X|} E_0^*} W = \underbrace{E_0^* W^* E_0 (W^*)^{-1}}_{=}$$

$$= f\, E_0^* E_0 (W^*)^{-1}$$

$$= \underbrace{\sum_{i=0}^{D} \tau_i^{-1} \underbrace{E_0^* E_0 E_i^*}_{|X|^{-1} E_0^* A_i}}_{=}$$

$$= |X|^{-1} E_0^* \left(\sum_{i=0}^{D} \tau_i^{-1} A_i \right)$$

so

$$0 = E_0^* \Big(\underbrace{f^{-1} \Big(\sum_{i=0}^{D} \tau_i^{-1} k_i\Big) W - \sum_{i=0}^{D} \tau_i^{-1} A_i}_{=:\, S} \Big)$$

$S \in M$ and $E_0^* S = 0$, row x of S is 0

so $S = 0$

(ii) Similar to (i). We have

$$W E_0^* W^{-1} = (W^*)^{-1} E_0 W^*$$

So

$$\underbrace{E_0^* W E_0^*}_{\displaystyle \| \atop \displaystyle \frac{f \sum_{i=0}^{D} \tau_i k_i}{|X|} E_0^*} W^{-1} = \underbrace{E_0^* (W^*)^{-1} E_0 W^*}$$

$$= f^{-1} E_0^* E_0 W^*$$

$$= \underbrace{\sum_{i=0}^{D} \tau_i \underbrace{E_0^* E_0 E_i^*}_{= |X|^{-1} E_0^* A_i}}$$

$$= |X|^{-1} E_0^* \left(\sum_{i=0}^{D} \tau_i A_i \right)$$

So

$$0 = E_0^* \Big(\underbrace{f \Big(\sum_{i=0}^{D} \tau_i k_i \Big) W - \sum_{i=0}^{D} \tau_i A_i}_{=: \Delta} \Big)$$

$\Delta \in M$ and $E_0^* \Delta = 0$ row x of Δ is 0

so $\Delta = 0$

(iii), (iv) Similar.

$\square$

Cor 35 With reference to Def 28,

(i) W and W^{-1} have all entries $\neq 0$.

(ii) $W \circ W^{-1} = |X|^{-1} J$

entrywise mult

J = all 1's matrix

pf By Prop 34 (i), (ii) $\square$

We next explain the significance of Cor 35

Def 36 For an integer $N \geq 1$ and an

$N \times N$ matrix w over $\mathbb{C}$,

w is called <u>type II</u> whenever

(i) w has all entries non 0

(ii) $w w^{(-)} \in \mathrm{Span}(I)$, where

$$w^{(-)}_{ij} = \frac{1}{w_{ji}} \qquad 1 \leq i,j \leq N$$

Ex $N = 2$

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \text{ non } 0$$

$$w^{(-)} = \begin{pmatrix} \frac{1}{a} & \frac{1}{c} \\ \frac{1}{b} & \frac{1}{d} \end{pmatrix}$$

$$w w^{(-)} = \begin{pmatrix} 2 & \frac{a}{c} + \frac{b}{d} \\ \frac{c}{a} + \frac{d}{b} & 2 \end{pmatrix}$$

W is type II iff both

$$\frac{c}{a} + \frac{d}{b} = 0, \qquad \frac{a}{c} + \frac{b}{d} = 0$$

iff

$$ad + bc = 0$$

In this case

$$W W^{(-)} = 2I$$

Ex For $N \geq 1$,

For an $N \times N$ matrix W over $\mathbb{C}$ that is type II

(i) $W W^{(-)} = N I$

(ii) W^{-1} exists

(iii) $W^{-1} = N^{-1} W^{(-)}$

(iv) $W^{t} \circ W^{-1} = N^{-1} J$

(v) W^{t} and W^{-1} are type II

By Cor 35, the matrix W from Def 28 is type II.

For W, W^* from Def 28, recall

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$$W W^* W = W^* W W^*$$

We now explore the meaning.

LEM 37 With reference to Def 28, for $y \in X$,

$$(W^*)_{yy} = \frac{\alpha}{W_{xy}} \qquad \alpha = \frac{f^2}{\sum_{i=0}^{D} \tau_i^{-1} k_i}$$

pf Recall

$$W^* = f \sum_{j=0}^{D} \tau_j E_j^*$$

$$W = f \frac{\sum_{j=0}^{D} \tau_j^{-1} A_j}{\sum_{j=0}^{D} \tau_j^{-1} k_j}$$

Let $\ell = \partial(x,y)$

$$W^*_{yy} = f \tau_\ell$$

$$W_{xy} = \frac{f \tau_\ell^{-1}}{\sum_{j=0}^{D} \tau_j^{-1} k_j} \qquad \text{ok}$$

$\square$

For $y, z \in X$ consider the (y,z)-entry of

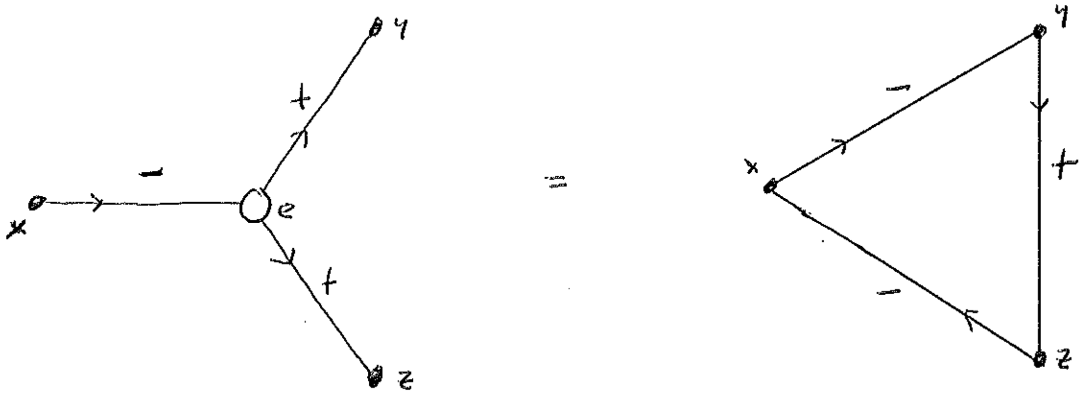
$$W W^* W = W^* W W^*$$

$$(W W^* W)_{yz} = \sum_{e \in X} \underbrace{W_{ye}}_{= W_{ey}} \underbrace{W^*_{ee}}_{= \frac{\alpha}{W_{xe}}} W_{ez}$$

$$(W^* W W^*)_{yz} = \underbrace{W^*_{yy}}_{= \frac{\alpha}{W_{xy}}} W_{yz} \underbrace{W^*_{zz}}_{= \frac{\alpha}{W_{zx}}}$$

So

$$\sum_{e \in X} \frac{W_{ey} W_{ez}}{W_{xe}} = \frac{W_{yz}}{W_{xy} W_{zx}} \alpha$$



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Note that $W = f \sum_{i=0}^{0} r_i E_i$ is indep

of the choice of x

Note also that $0 \neq f \in \mathbb{C}$ is free so $0 \neq \alpha \in \mathbb{C}$

is free. It is convenient to choose

f s.t. $\alpha = |X|^{1/2}$

Def 38 For an integer $N \geq 1$ and an $N \times N$ matrix W over $\mathbb{C}$,

W is a <u>spin model</u> whenever

(i) W is type II

(ii) For $1 \leq x, y, z \leq N$

$$\sum_{e=1}^{N} \frac{W_{ey} W_{ez}}{W_{xe}} = \frac{W_{yz}}{W_{xy} W_{zx}} N^{1/2}$$

"type III condition" "star-triangle condition"

Prop 39 The matrix W from Def 28 is a spin model, provided that the scalar f is chosen st

$$f^2 = N^{1/2} \sum_{i=0}^{D} \tau_i^{-1} k_i$$

Pf By the discussion above Def 38 $\square$

Spin model reference:

F. Jaeger, M. Matsumoto, K. Nomura

Bose-Mesner algebras related to type II

matrices and spin models,

J. Algebraic Combinatorics 8 (1998) 39-72

F. Jaeger.

On spin models, triply-regular association

schemes, and duality

J. Algebraic Combinatorics 4 (1995) 103-144

Exercise IV Assume $\Gamma = (X, R)$ is the complete graph K_n $n \geq 2$ $\mathbb{F} = \mathbb{C}$.

Given $W \in M$ so W has form

$$W = \begin{pmatrix} a & & & b \\ & a & & \\ & & \ddots & \\ b & & & a \end{pmatrix} \qquad a, b \in \mathbb{C}.$$

(i) Find nec/suf conditions on a, b st W has type II.

(ii) Find nec/suf conditions on a, b st W is a spin model