

Combinatorics 475

Lecture 24

7.4 Linear homogeneous Recurrence Relations Lec 24

Given a sequence $\{h_n\}_{n=0}^{\infty}$ of real numbers

suppose it satisfies a recurrence such as

$$h_n = a h_{n-1} + b h_{n-2} + c \quad n \geq 2 \quad *$$

Possibly a, b, c depend on n

Ex Fibonacci numbers

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2$$

$$f_0 = 0, \quad f_1 = 1$$

Ex Derangement numbers

View I:

$$D_n = (n-1) D_{n-1} + (n-1) D_{n-2} \quad n \geq 2$$

$$D_0 = 1, \quad D_1 = 0$$

View II:

$$D_n = n D_{n-1} + (-1)^n \quad n \geq 1$$

$$D_0 = 1$$

Special cases of *

- a, b, c indep of n

"constant coefficients"

- $c = 0$

"homogeneous"

Until further notice assume both
constant coef + homog

Ex Solve the recurrence

$$h_n = 3h_{n-1} + 4h_{n-2} - 12h_{n-3} \quad n \geq 3$$

*

subject to

$$h_0 = 5,$$

$$h_1 = -11,$$

$$h_2 = 15$$

**

Sol First ignore ** and find gen sol to *

Hunt for sol of form

$$h_n = g^n$$

$$n \geq 0$$

Require

$$q^n = 3q^{n-1} + 4q^{n-2} - 12q^{n-3} \quad n \geq 3$$

Take $n=3$

$$q^3 = 3q^2 + 4q - 12$$

or

$$q^3 - 3q^2 - 4q + 12 = 0$$

So q is a root of polynomial

$$x^3 - 3x^2 - 4x + 12$$

"characteristic poly"

Factor

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 &= (x^2 - 4)(x - 3) \\ &= (x - 2)(x + 2)(x - 3) \end{aligned}$$

$$q \in \{2, -2, 3\}$$

So far we found 3 sols

Sol 1

$$h_n = 2^n$$

Sol 2

$$h_n = (-2)^n$$

Sol 3

$$h_n = 3^n$$

We combine above 3 sols to get
general sol:

For arbitrary real numbers a, b, c

$$h_n = a 2^n + b (-2)^n + c 3^n$$

$$n = 0, 1, 2, \dots$$



is a sol to *

claim

Any sol to * has form *

pt claim

Call the sol $\{H_n\}_{n=0}^{\infty}$

Consider H_0, H_1, H_2

Solve the equations

for a, b, c

$$a + b + c = H_0$$

$$2a - 2b + 3c = H_1$$

$$2^2 a + (-2)^2 b + 3^2 c = H_2$$

for a, b, c . Using these a, b, c define

$$h_n = a 2^n + b (-2)^n + c 3^n$$

$$n = 0, 1, 2, \dots$$

We have

• $\{h_n\}_{n=0}^{\infty}$ is sol to *

• $\{H_n\}_{n=0}^{\infty}$ is sol to *

• $H_n = h_n$ for $n = 0, 1, 2$

So $H_n = h_n$ for $n \geq 0$

claim proved.

So far

$$h_n = a2^n + b(-2)^n + c3^n \quad n=0,1,2,\dots$$

Now consider init cond **

Require

$$n=0: \quad a + b + c = 5$$

$$n=1: \quad 2a - 2b + 3c = -11$$

$$n=2: \quad 4a + 4b + 9c = 15$$

solving this system (ex) method

$$a=1 \quad b=5 \quad c=-1$$

So -

$$h_n = 2^n + 5(-2)^n - 3^n \quad n \geq 0$$

□

— 0 —
Above method works in general as long as
char poly has all roots distinct

We now consider when char poly has repeated roots

7.4 cont

Summary

solve the rec

$$h_n = 3h_{n-1} + 4h_{n-2} - 12h_{n-3}$$

$$n \geq 3$$

*

subject to

$$h_0 = 5$$

$$h_1 = -11$$

$$h_2 = 15$$

**

Sol skps

- Factor char poly

$$X^3 - 3X^2 - 4X + 12 = (X-2)(X+2)(X-3)$$

- gen sol to * is

$$h_n = a2^n + b(-2)^n + c3^n \quad n=0,1,2$$

- Use ** to find

$$a=1, \quad b=5, \quad c=-7$$

Sol is

$$h_n = 2^n + 5(-2)^n - 3^n \quad n=0,1,2,\dots$$

Above method works provided the char poly has no repeated roots.

We now consider case of repeated roots.

ex Solve the recurrence

$$h_n = 6h_{n-1} - 12h_{n-2} + 8h_{n-3} \quad n \geq 3$$

subject to

$$h_0 = 5,$$

$$h_1 = 0,$$

$$h_2 = -12$$

**

Sol Factor char poly

$$X^3 - 6X^2 + 12X - 8$$

$$= X^3 - 3 \cdot X^2 \cdot 2 + 3 \cdot X \cdot 2^2 - 2^3$$

$$= (X - 2)^3$$

Repeated root 2, 2, 2.

claim Each of the following is a sol to * :

(i) $h_n = 2^n$

(ii) $h_n = n 2^n$

(iii) $h_n = n^2 2^n$

pt cl

(i) as before

(ii)

$$n2^n = ? \quad 6(n-1)2^{n-1} - 12(n-2)2^{n-2} + 8(n-3)2^{n-3}$$

divide by 2^n

$$n = ? \quad 3(n-1) - 3(n-2) + n-3$$

✓

(iii) similar

claim proved.

Now the general solution to * is

$$h_n = a2^n + b n 2^n + c n^2 2^n$$

 $n = 0, 1, 2, \dots$ To find a, b, c use ** $n=0:$

$$a = 5$$

 $n=1:$

$$a2 + b2 + c \cdot 2 = 0$$

 $n=2:$

$$a4 + b8 + c16 = -12$$

Get

$$a = 5$$

$$b = -6$$

$$c = 1$$

$$\text{So } h_n = (5 - 6n + n^2)2^n$$

$$= (n-5)(n-1)2^n$$

 $n = 0, 1, 2, \dots$

□

Ex Solve the recurrence

$$h_n = 2h_{n-1} + 4h_{n-2} - 8h_{n-3} \quad n \geq 3$$

subject to

$$h_0 = -2,$$

$$h_1 = -6$$

$$h_2 = 0$$

Sol Factor char poly

$$\begin{aligned} x^3 - 2x^2 - 4x + 8 &= (x^2 - 4)(x - 2) \\ &= (x + 2)(x - 2)^2 \end{aligned}$$

Roots are

$$2, 2, -2$$

Each of the following is sol to *:

$$h_n = 2^n$$

$$h_n = n2^n$$

$$h_n = (-2)^n$$

Gen sol to * is

$$h_n = (a + bn)2^n + c(-2)^n$$

$$n = 0, 1, 2, \dots$$

Use $**$ to solve for a, b, c

Get

$$a = -3,$$

$$b = 1,$$

$$c = 1$$

Sol to $*, **$ is

$$h_n = (n-3)2^n + (-2)^n$$

$$n = 0, 1, 2, \dots$$

□

We now show how to use gen functions
to solve a linear recurrence

ex Solve the recurrence

$$h_n = 3h_{n-2} - 2h_{n-3}$$

$$n \geq 3$$

$$h_0 = 1, \quad h_1 = 0, \quad h_2 = 0$$

Sol Consider gen function

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

Obs

$$\sum_{n=2}^{\infty} h_{n-2} x^n = \sum_{n=0}^{\infty} h_n x^{n+2} = x^2 g(x)$$

$$\sum_{n=3}^{\infty} h_{n-3} x^n = \sum_{n=0}^{\infty} h_n x^{n+3} = x^3 g(x)$$

$$g(x) - 3x^2 g(x) + 2x^3 g(x) =$$

$$h_0 + h_1 x + h_2 x^2 - 3(h_0 x^2)$$

$$= 1 - 3x^2$$

$$g(x) \left(1 - 3x^2 + 2x^3 \right) = 1 - 3x^2$$

So

$$g(x) = \frac{1 - 3x^2}{1 - 3x^2 + 2x^3}$$

factor denom

$$= \frac{1 - 3x^2}{(1 - x)^2 (1 + 2x)}$$

partial fraction decomp

$$= \frac{8}{9} \frac{1}{(1-x)^2} - \frac{14}{9} \frac{x}{(1-x)^2} + \frac{1}{9} \frac{1}{1+2x}$$

$$= \frac{8}{9} \sum_{n=0}^{\infty} (n+1)x^n - \frac{14}{9} \sum_{n=0}^{\infty} nx^n + \frac{1}{9} \sum_{n=0}^{\infty} (-2)^n x^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{8(n+1) - 14n + (-2)^n}{9} \right) x^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{8 - 6n + (-2)^n}{9}}_{h_n} x^n$$

$$h_n = \frac{8 - 6n + (-2)^n}{9}$$

 $n = 0, 1, 2, \dots$

□

