Math 846

Lecture 24

Remarks

Given any DRG $\Gamma = (X, R)$ diam D

By Lem 2 in CH 1 TFAE

(i) $b_i = c_{D-i}$ for $0 \le i \le D$

(ii) $k_i = k_{D-i}$ for $0 \le i \le D$

(iii) $k_D = 1$.

Γ is called an antipodal 2-cover whenever (i) – (iii) hold.

Each Γ in Lem 24, 25, 26 is an antipodal 2-cover.

Until further notice we consider the DRG Γ from Lem 25.

Referring to Lem 25, we have

$$\beta = q^2 + q^{-2},$$

$$\gamma = \gamma^* = 0,$$

$$\delta = \delta^* = (q^{D-2} + q^{2-D})^2.$$

Fix $x \in X$ and write $A^* = A^*(x)$.
By Th 19,

$$A^2 A^* - (q^2+q^{-2}) A A^* A + A^* A^2 = (q^{D-2} + q^{2-D})^2 A^*,$$

$$A^{*2} A - (q^2+q^{-2}) A^* A A^* + A A^{*2} = (q^{D-2} + q^{2-D})^2 A$$

In cyclic form this looks as follows:

Abbreviate $B = A^*$.

LEM 27 With reference to Lem 25 Lec 24

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assume $\mathbb{F} = \mathbb{C}$. Then $\exists$ $C \in Mat_X(\mathbb{F})$

st

$$qAB - q^{-1}BA = zC,$$

$$qBC - q^{-1}CB = zA,$$

$$qCA - q^{-1}AC = zB$$

where

$$z = i(q^{D-2} + q^{2-D}) \qquad i^2 = -1$$

pf Routine. $\square$

In what follows we refer to the matrices A, B, C in Lem 27

Next goal: find the eigenvalues of the matrix C.

Strategy: Search for $W \in M$ and $W^* \in M^*$

st
$$WBW^{-1} = C, \qquad (W^*)^{-1} A W^* = C.$$

Assume for the moment that W exists.

Write
$$W = \sum_{i=0}^{D} \alpha_i E_i \qquad \alpha_i \in \mathbb{C}$$

W^{-1} exists so $\alpha_i \neq 0$ $(0 \le i \le D)$ and
$$W^{-1} = \sum_{i=0}^{D} \alpha_i^{-1} E_i$$

Require

$$C = W B W^{-1}$$

For $0 \le i, j \le D$ require

$$E_i C E_j = E_i (W B W^{-1}) E_j$$

$$= E_i B E_j \, \alpha_i / \alpha_j$$

We have

$$E_i C E_j = E_i \frac{q A B - q^{-1} B A}{z} E_j$$

$$= E_i B E_j \frac{q \theta_i - q^{-1} \theta_j}{z}$$

Since our Q-poly structure is dual bipartite,

$$E_i B E_j = 0 \text{ iff } |i-j| \neq 1$$

Require

$$\alpha_i / \alpha_j = \frac{q \theta_i - q^{-1} \theta_j}{z} \quad \text{if } |i-j| = 1 \quad (0 \le i, j \le D)$$

Recall for $1 \le i \le D$,

$$\underset{\underset{-z^2}{\parallel}}{\delta} = \theta_{i-1}^2 - \underset{\underset{q^2+q^{-2}}{\parallel}}{\beta}\, \theta_{i-1}\theta_i + \theta_i^2 - \underset{\underset{0}{\parallel}}{\gamma}\,(\theta_{i-1} + \theta_i)$$

$$-z^2 = \left(q\theta_{i-1} - q^{-1}\theta_i\right)\left(q^{-1}\theta_{i-1} - q\theta_i\right)$$

So

$$1 = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \quad \frac{q\theta_{i-1} - q^{-1}\theta_i}{z}$$

So for $0 \le i, j \le D$

$$1 = \frac{q\theta_i - q^{-1}\theta_j}{z} \quad \frac{q\theta_j - q^{-1}\theta_i}{z} \qquad \text{if} \quad |i-j| = 1$$

Our requirement on $\{\alpha_i\}_{i=0}^{D}$ is just

$$\frac{\alpha_i}{\alpha_{i-1}} = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \qquad (1 \le i \le D)$$

By Lem 25

$$\frac{q\theta_i - q^{-1}\theta_{i-1}}{z} = \iota^{0}\, q^{2i-1-D} \qquad (1 \le i \le D)$$

$$\left((\iota^{0})^2 = -1\right)$$

the requirement on $\{\alpha_i\}_{i=0}^{D}$ is

$$\alpha_0 \neq 0, \qquad \alpha_i/\alpha_{i-1} = \iota^{0} q^{2i-1-D} \quad (1 \le i \le D)$$

We make a change of variables:

$$f = \alpha_0 \qquad \tau_i = \alpha_i/\alpha_0 \quad (0 \le i \le D)$$

so $\tau_0 = 1$ and $\alpha_i = f\tau_i \quad (0 \le i \le D)$

We now officially define W and W^*

DEF 28 Pick $0 \neq f \in \mathbb{C}$. Define scalars $\{\tau_i\}_{i=0}^{D}$ by

$$\tau_0 = 1, \qquad \tau_i/\tau_{i-1} = \iota^{0} q^{2i-D-1} \quad (1 \le i \le D)$$

Define

$$W = f \sum_{i=0}^{D} \tau_i E_i, \qquad W^* = f \sum_{i=0}^{D} \tau_i E_i^*$$

LEM 29 With above notation,

$$W B W^{-1} = C, \qquad (W^*)^{-1} A W^* = C$$

pf By the argument above Def 28 $\square$

COR 30 The matrix C is diagonalizable, with distinct eigenvalues

$$\theta_i = \theta_i^* \qquad (0 \le i \le D)$$

pf By Lem 29 and linear algebra.

LEM 31 Define the map

$$\rho : \begin{array}{ccc} T & \longrightarrow & T \\ m & \longrightarrow & (WW^*)\, m\, (WW^*)^{-1} \end{array}$$

and observe that ρ is an algebra isomorphism.

The map ρ sends

$$A \to B \to C \to A$$

pf show ρ sends $A \to B$:

$$WW^*A \overset{?}{=} BWW^*$$

Given

$$WBW^{-1} = C = (W^*)^{-1} A W^*$$

so

$$W^*WB = AW^*W$$

take transpose

$$BWW^* = WW^*A \qquad \text{ok}$$

check ρ sends $B \to C$:

$$ww^* B \overset{?}{=} C ww^*$$
$$C = \underbrace{WBW^{-1}}_{= WBW^*}$$
$$W \underbrace{BW^*}_{= W^*B} \quad \text{since } B, W^* \in M^*$$

ok

check ρ sends $C \to A$:

$$ww^* C \overset{?}{=} A ww^*$$
$$C = \underbrace{(W^*)^{-1} A W^*}_{= WAW^*}$$
$$\underbrace{WA}_{= AW} W^*$$

ok

$\square$

LEM 32 We have

(i) $W E_i^* W^{-1} = (W^*)^{-1} E_i W^* \quad (0 \le i \le D)$

(ii) $W^{-1} E_i^* W = W^* E_i (W^*)^{-1} \quad (0 \le i \le D)$

(iii) $W W^* W = W^* W W^*$ "Braid relation"

pf (i) Recall

$$W B W^{-1} = (W^*)^{-1} A W^*,$$

$$E_i = \prod_{\substack{0 \le j \le D \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \qquad E_i^* = \prod_{\substack{0 \le j \le D \\ j \ne i}} \frac{B - \theta_j^* I}{\theta_i^* - \theta_j^*}$$

$$\theta_\ell = \theta_\ell^* \quad (0 \le \ell \le D)$$

Result follows.

(ii) In (i) take the transpose of each side

(iii) Obs

$$W W^* W^{-1} = f \sum_{i=0}^{D} \tau_i \, W E_i^* W^{-1}$$

$$= f \sum_{i=0}^{D} \tau_i (W^*)^{-1} E_i W^*$$

$$= (W^*)^{-1} W W^*$$

So $W^* W W^* = W W^* W$ $\square$

LEM 33 With reference to Def 28,

(i) $$E_0^* W E_0^* = \frac{f \sum_{i=0}^{D} r_i k_i}{|X|} E_0^*$$

(ii) $$E_0^* W^{-1} E_0^* = \frac{\sum_{i=0}^{D} r_i^{-1} k_i}{f |X|} E_0^*$$

(iii) $$E_0 W^* E_0 = \frac{f \sum_{i=0}^{D} r_i k_i}{|X|} E_0$$

(iv) $$E_0 (W^*)^{-1} E_0 = \frac{\sum_{i=0}^{D} r_i^{-1} k_i}{f |X|} E_0$$

(v) $$|X| = \left(\sum_{i=0}^{D} r_i k_i \right) \left(\sum_{i=0}^{D} r_i^{-1} k_i \right)$$

(vi) $$\sum_{i=0}^{D} r_i k_i \neq 0, \qquad \sum_{i=0}^{D} r_i^{-1} k_i \neq 0$$

pf (i) obs

$$E_0^* W E_0^* = f \sum_{i=0}^{D} \tau_i \underbrace{E_0^* E_i E_0^*}_{|X|^{-1} k_i E_0^*}$$

$$= |X|^{-1} f \left(\sum_{i=0}^{D} \tau_i k_i \right) E_0^*$$

(ii) obs

$$E_0^* W^{-1} E_0^* = f^{-1} \sum_{i=0}^{D} \tau_i^{-1} \underbrace{E_0^* E_i E_0^*}_{|X|^{-1} k_i E_0^*}$$

$$= |X|^{-1} f^{-1} \left(\sum_{i=0}^{D} \tau_i^{-1} k_i \right) E_0^*$$

(iii), (iv) Similar

(v) We have

$$W E_0^* W^{-1} = (W^*)^{-1} E_0 W^*$$

So

$$E_0^* W E_0^* W^{-1} E_0^*$$

$$= \underbrace{E_0^* (W^*)^{-1} E_0}_{= \rho^{-1} E_0^*} \underbrace{W^* E_0^*}_{= \rho E_0^*}$$

$$= E_0^* E_0 E_0^*$$

$$= |X|^{-1} E_0^*$$

Simplify the above eqn using (i), (ii) to get

$$|X|^{-2} \left(\sum_{i=0}^{D} \tau_i k_i \right) \left(\sum_{i=0}^{D} \tau_i^{-1} k_i \right) E_0^* = |X|^{-1} E_0^*$$

Result follows.

(vi) By (v). $\square$