

Combinatorics 475

Lecture 23

7.3 Exponential Generating Functions

Given a sequence of real numbers

$$\{h_n\}_{n=0}^{\infty}$$

then exponential generating function is

$$g^e(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

ex

$$h_n = 1$$

$$n = 0, 1, 2, \dots$$

$$g^e(x) = \sum_{n=0}^{\infty} 1 \frac{x^n}{n!}$$

$$= e^x$$

Ex Fix integer $m \geq 0$

For $0 \leq n \leq m$ let

$$h_n = \# \text{ } n\text{-perms of } \{1, 2, \dots, m\}$$

Find exp gen function

Sol Recall

$$\begin{aligned} h_n &= P(m, n) \\ &= m(m-1) \cdots (m-n+1) \end{aligned}$$

$$g^e(x) = \sum_{n=0}^m P(m, n) \frac{x^n}{n!}$$

$$= \sum_{n=0}^m \binom{m}{n} x^n$$

$$= (1+x)^m$$

□

Ex Fix integer $k \geq 1$

Consider multiset

$$\{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$$

*

each $n_i = \text{pos integer or } \infty$

For $n \geq 0$ let

$$h_n = \# \text{ } n\text{-perms of } *$$

Find exp gen function for $\{h_n\}_{n=0}^{\infty}$

Sol

We get an n -perm of $*$ in two stages

stage 1

Pick an n -combination of $*$:

$$\{b_1 \cdot a_1, b_2 \cdot a_2, \dots, b_k \cdot a_k\}$$

$$0 \leq b_1 \leq n_1$$

$$0 \leq b_2 \leq n_2$$

$\vdots$

$$0 \leq b_k \leq n_k$$

$$b_1 + b_2 + \dots + b_k = n$$

Stage 2 order the elements of the n -comb

ways is

$$\frac{n!}{b_1! b_2! \dots b_k!}$$

So

$$h_n = \# \text{ } n\text{-perms of } *$$

$$= \sum \frac{n!}{b_1! b_2! \dots b_k!}$$

↑

sum over all sequences $b_1, b_2, \dots, b_k$ such that

$$0 \leq b_1 \leq n_1$$

$$0 \leq b_2 \leq n_2$$

⋮

$$0 \leq b_k \leq n_k$$

$$b_1 + b_2 + \dots + b_k = n$$

Now

$$g^c(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\substack{0 \leq b_i \leq n_i \\ b_1 + \dots + b_k = n}} \frac{\cancel{n!}}{b_1! b_2! \dots b_k!} \right) \frac{x^n}{\cancel{n!}} = x^{b_1 + b_2 + \dots + b_k}$$

$$= \sum_{b_1=0}^{n_1} \sum_{b_2=0}^{n_2} \dots \sum_{b_k=0}^{n_k} \frac{x^{b_1 + b_2 + \dots + b_k}}{b_1! b_2! \dots b_k!}$$

$$= \left(\sum_{b_1=0}^{n_1} \frac{x^{b_1}}{b_1!} \right) \left(\sum_{b_2=0}^{n_2} \frac{x^{b_2}}{b_2!} \right) \dots \left(\sum_{b_k=0}^{n_k} \frac{x^{b_k}}{b_k!} \right)$$

$$\left[\begin{array}{l} \text{define} \\ f_m(x) = \begin{cases} 1 + x + \frac{x^2}{2!} + \dots + \frac{x^m}{m!} & \text{if } m < \infty \\ e^x & \text{if } m = \infty \end{cases} \end{array} \right]$$

$$= f_{n_1}(x) f_{n_2}(x) \dots f_{n_k}(x)$$

□

Ex For $n \geq 0$ define

$h_n = \#$ n -digit numbers using digits 1, 2, 3
with 1 used an even number of times

Find exp gen function

Sol Obs

$h_n = \#$ n -perms of

$$\{ \infty \cdot 1, \infty \cdot 2, \infty \cdot 3 \}$$

such that 1 is used an even number of times

As in previous example,

$$g^e(x) =$$

$$\left(\underbrace{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots}_{\parallel e^x} \right) \left(\underbrace{1 + x + \frac{x^2}{2!} + \dots}_{\parallel e^x} \right) \left(\underbrace{1 + x + \frac{x^2}{2!} + \dots}_{\parallel e^x} \right)$$

$$\frac{e^x + e^{-x}}{2}$$

$$= \frac{e^{3x} + e^x}{2}$$

$$= \sum_{n=0}^{\infty} \left(\frac{3^n + 1}{2} \right) \frac{x^n}{n!}$$

$$\text{So } h_n = \frac{3^n + 1}{2} \quad n = 0, 1, 2, \dots$$

□

Ex for $n \geq 0$ let

$h_n = \#$ ways to color a $1 \times n$ chessboard with colors

Red, White, Blue

such that

- number of red squares is even
- at least one blue square

Find h_n

Sol, Compute exp gen function

Obs

$h_n = \#$ n-perms of

$$\{ \infty = R, \infty = W, \infty = B \}$$

such that

- R is used an even number of times
- B used at least once

$$g^e(x) =$$

$$\left(\begin{array}{ccc} 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots & \times & 1 + x + \frac{x^2}{2!} + \dots \\ & & \times \\ & & x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{array} \right)$$

$\parallel$ $\parallel$ $\parallel$
 $e^x + e^{-x}$ e^x $e^x - 1$
 $\left\{ \frac{e^x + e^{-x}}{2} \right.$

$$= \frac{e^{3x} - e^{2x} + e^x - 1}{2}$$

$$= -\frac{1}{2} + \sum_{n=0}^{\infty} \left(\frac{3^n - 2^n + 1}{2} \right) \frac{x^n}{n!}$$

$\uparrow$
 for $n=0$ term is $\frac{1-1+1}{2} = \frac{1}{2}$

$$= \sum_{n=1}^{\infty} \frac{3^n - 2^n + 1}{2} \frac{x^n}{n!}$$

so

$$h_n = \begin{cases} \frac{3^n - 2^n + 1}{2} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

□

Ex For $n \geq 0$ let h_n denote the number of n -permutations of the multiset

$$\{\infty \cdot 1, \infty \cdot 3, \infty \cdot 5, \infty \cdot 7, \infty \cdot 9\}$$

such that each of $1, 3, 5, 7, 9$ appears an odd number of times. Find h_n .

Sol The exponential generating function is

$$g^e(x) = \sum_{n=0}^{\infty} \frac{h_n x^n}{n!}$$

$$= G_1(x) G_3(x) G_5(x) G_7(x) G_9(x)$$

where

$$G_1(x) = G_3(x) = G_5(x) = G_7(x) = G_9(x)$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$= \frac{e^x - e^{-x}}{2}$$

Using this we obtain

$$g^e(x) = \frac{(e^x - e^{-x})^5}{2^5}$$

$$= \frac{e^{5x} - 5e^{3x} + 10e^x - 10e^{-x} + 5e^{-3x} - e^{-5x}}{2^5}$$

$$= \sum_{n=0}^{\infty} \frac{5^n - 5 \cdot 3^n + 10 \cdot 1^n - 10 \cdot (-1)^n + 5 \cdot (-3)^n - (-5)^n}{32} \frac{x^n}{n!}$$

Therefore

$$h_n = \frac{5^n - 5 \cdot 3^n + 10 \cdot 1^n - 10(-1)^n + 5(-3)^n - (-5)^n}{32}$$

$$\text{for } n = 0, 1, 2, \dots$$

□

