Math 846

Lecture 23

LEM 23 Assume Γ is bipartite, and the Q-poly ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite. Further assume $\beta = 2$. Then

(i) $\theta_i = \theta_i^* = D - 2i$ $(0 \le i \le D)$

(ii) $c_i = c_i^* = i$ $(0 \le i \le D)$

(iii) $b_i = b_i^* = D - i$ $(0 \le i \le D)$

pf Assume $D \ge 2$; else trivial.

We are in Case II: the θ_i and θ_i^* have form

$$\theta_i = a + bi + ci^2 \qquad 0 \le i \le D$$
$$\theta_i^* = a^* + b^* i + c^* i^2$$

By Th.19 (i)

$$0 = \gamma = \theta_0 - \beta\theta_1 + \theta_2$$
$$= a - 2(a+b+c) + a + 2b + 4c$$
$$= 2c$$

So $c = 0$

Similarly $c^* = 0$

The constraint

$$\frac{\theta_{D-1}}{\theta_D} = \frac{\theta_1}{\theta_0}$$

Gives $$b/a = \frac{-2}{D}$$

Similarly $$b^*/a^* = -2/D$$

So far, $$\frac{\theta_i}{\theta_0} = 1 - \frac{2i}{D} \qquad 0 \le i \le D$$

$$\frac{\theta_i^*}{\theta_0^*} = 1 - \frac{2i}{D} \qquad 0 \le i \le D$$

For $1 \le i \le D-1$ solve for c_i using LEM 22 (i)

to get $$c_i = \frac{i \theta_0}{D}$$

But $c_1 = 1$ so

$$\theta_0 = D$$

Hence $$\theta_i = D - 2i \qquad (0 \le i \le D)$$

Also

$$c_n = a_0 = n$$

So

$$c_i = i \qquad (1 \le i \le n-1)$$

Now

$$c_i = i \qquad (0 \le i \le n)$$

Obs

$$b_i = k - c_i \qquad k = a_0 = n$$

$$= n - i$$

for $0 \le i \le n$.

Similarly

$$a_i^r = n - 2i$$

$$c_i^r = i$$

$$b_i^r = n - i$$

for $0 \le i \le n$. $\square$

LEM 24 For a DRG $\Gamma = (X, R)$ with diameter D, TFAE:

(i) Γ is bipartite and

$$c_i = i; \qquad b_i = D-i \qquad (0 \le i \le D)$$

(ii) Γ is the hypercube $H(D,2)$

pf (i) $\to$ (ii) Assume $D \ge 1$, else trivial.

Γ has same c_i, b_i as $H(D,2)$, so Γ has same eigenvalues and Krein parameters as $H(D,2)$

Γ has eigenvalues

$$\theta_i = D - 2i \qquad (0 \le i \le D)$$

Corresp $\{E_i\}_{i=0}^{D}$ is dual bipartite Q-poly.

$E = E_1$ has dual eigenvalues

$$\theta_i^* = D - 2i \qquad (0 \le i \le D)$$

$$\dim EV = \theta_0^* = D$$

Recall, $\forall x, y \in X$

$$\langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta_i^* \qquad i = \partial(x,y) \qquad (*)$$

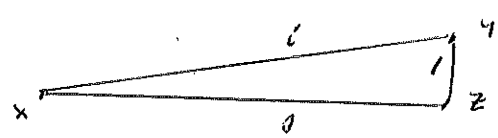
Consider the set Φ of distinct vectors among

$$\frac{E\hat{x} - E\hat{y}}{2} \qquad x, y \in X, \qquad \partial(x,y) = 1$$

- $r \in \Phi$ implies $-r \in \Phi$
- By $(*)$,
$$\|r\|^2 = |X|^{-1} \qquad r \in \Phi$$
- For $r \in \Phi$ and $x \in X$,
$$|X| \langle r, E\hat{x} \rangle \in \{1, -1\}$$

pf: Write
$$r = \frac{E\hat{y} - E\hat{z}}{2} \qquad y, z \in X, \qquad \partial(y,z) = 1$$

Define
$$i = \partial(x,y), \qquad j = \partial(x,z)$$



By the triangle inequality and since Γ is bipartite,

$$|i-j| = 1$$

Obs

$$|X| \langle r, E\hat{x} \rangle = |X| \left\langle \frac{E\hat{y} - E\hat{z}}{2}, \; E\hat{x} \right\rangle$$

$$= \frac{\theta_i^* - \theta_j^*}{2}$$

$$= \frac{p - 2i - p + 2j}{2}$$

$$= j - i$$

$$\in \{1, -1\} \checkmark$$

- For $r, s \in \Phi$ either

$r = s$ or $r = -s$ or $\langle r, s \rangle = 0$

pf: Write

$$s = \frac{E\hat{x} - E\hat{y}}{2}, \qquad x, y \in X, \qquad \partial(x,y) = 1$$

Obs

$$|X| \langle r, s \rangle = |X| \, \frac{\langle r, E\hat{x} \rangle - \langle r, E\hat{y} \rangle}{2}$$

$$= \frac{(\pm 1) - (\pm 1)}{2}$$

$$\in \{1, 0, -1\}$$

the cosine

$$\frac{\langle r, s \rangle}{\|r\| \, \|s\|} = |X| \langle r, s \rangle \in \{1, 0, -1\}$$

So $r = \pm s$ or $\langle r, s \rangle = 0$

- $EV = \mathrm{Span}(\Phi)$

pf: For $x \in X$,

$$\sum_{y \in \Gamma(x)} E\hat{y} = \theta_1 E\hat{x} \qquad \theta_1 = D - 2$$

So

$$E\hat{x} = \sum_{y \in \Gamma(x)} \frac{E\hat{x} - E\hat{y}}{2} \in \Phi$$

So

$$EV = \mathrm{Span}(E\hat{x} \mid x \in X) = \mathrm{Span}(\Phi) \quad \checkmark$$

- We have

$$\Phi = \{\pm e_1, \pm e_2, \ldots, \pm e_D\}$$

where

$$\|e_i\|^2 = |X|^{-1} \qquad (1 \le i \le D)$$

$$\langle e_i, e_j \rangle = 0 \quad \text{if } i \ne j \qquad (1 \le i, j \le D)$$

For $1 \le i \le p$ and $x \in X$, define

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$$\varepsilon_i(x) = |X| \langle e_i, E\hat{x} \rangle$$

Obs $\qquad \varepsilon_i(x) \in \{1, -1\}$

- For $x \in X$, $\qquad$ (**)

$$E\hat{x} = \sum_{i=1}^{p} \varepsilon_i(x) e_i$$

pf: the $\{e_i\}_{i=1}^{p}$ form a basis for EV, so

$$E\hat{x} = \sum_{i=1}^{p} a_i e_i \qquad a_i \in \mathbb{F}$$

For $1 \le j \le p$,

$$\langle e_j, E\hat{x} \rangle = \langle e_j, \sum_{i=1}^{p} a_i e_i \rangle$$

$$= \bar{a}_j \underbrace{\| e_j \|^2}_{= |X|^{-1}}$$

So
$$a_j = \bar{a}_j = |X| \langle e_j, E\hat{x} \rangle = \varepsilon_j(x) \quad \checkmark$$

The vertex set $\underline{X}$ of $H(D,2)$ consists of the D-tuples

$$(a_1, a_2, \ldots, a_D) \qquad a_i \in \{1, -1\} \quad (1 \le i \le D)$$

Define a function

$$\mathcal{E} : \begin{array}{ccc} X & \longrightarrow & \underline{X} \\ x & \longrightarrow & (\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_D(x)) \end{array}$$

Show $\mathcal{E}$ is an isomorphism of graphs $\Gamma \to H(D,2)$

First, show $\mathcal{E}$ is a bijection.

check $\mathcal{E}$ is injective:

We have $\theta_i^* \ne \theta_D^*$ $(1 \le i \le D)$

So E is nondegenerate

So $\{E\hat{x}\}_{x \in X}$ are mutually distinct.

By this and $(**)$,

$\mathcal{E}$ is injective.

Also $|X| = 2^D = |\underline{X}|$

So $\mathcal{E}$ is bijection.

Show E respects path-length distance Lec $^{23}_{10}$

Recall ∂ is path-length distance function for Γ.

For $x, y \in X$ show

$$\underbrace{\partial(x,y)}_{\overset{\text{def}}{=}\,\ell} = \underbrace{\left|\{ i \mid 1 \le i \le D,\ \varepsilon_i(x) \ne \varepsilon_i(y) \}\right|}_{\overset{\text{def}}{=}\,h}$$

obs

$$\begin{aligned}
D - 2\ell &= \theta^*_\ell \\
&= |X| \langle E\hat{x}, E\hat{y} \rangle \\
&= |X| \left\langle \sum_{i=1}^{D} \varepsilon_i(x) e_i,\ \sum_{j=1}^{D} \varepsilon_j(y) e_j \right\rangle \\
&= |X| \sum_{i=1}^{D} \varepsilon_i(x)\varepsilon_i(y) \underbrace{\|e_i\|^2}_{=\,|X|^{-1}} \\
&= \sum_{i=1}^{D} \varepsilon_i(x)\varepsilon_i(y) \\
&= \overbrace{\left|\{ i \mid 1 \le i \le D,\ \varepsilon_i(x) = \varepsilon_i(y) \}\right|}^{=\,D-h} \\
&\qquad - \underbrace{\left|\{ i \mid 1 \le i \le D,\ \varepsilon_i(x) \ne \varepsilon_i(y) \}\right|}_{=\,h} \\
&= D - 2h
\end{aligned}$$

So $\ell = h$ ✓ Done

$(ii) \to (i)$ clear $\square$

LEM 25 Assume Γ is bipartite, and the Q-poly ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite. Further assume $\beta \neq \mp 2$. Write $\beta = q^2 + q^{-2}$, $0 \neq q \in \mathbb{C}$. Then Lec 11

(i) $\theta_i = \theta_i^* =$

$$\left(q^{D-2} + q^{2-D}\right) \frac{q^{D-2i} - q^{2i-D}}{q^2 - q^{-2}} \qquad (0 \le i \le D)$$

(ii) $c_i = c_i^* =$

$$\frac{q^{D-2} + q^{2-D}}{q^{D-2i} + q^{2i-D}} \frac{q^{2i} - q^{-2i}}{q^2 - q^{-2}} \qquad (1 \le i \le D-1)$$

$$c_D = c_D^* = \left(q^{D-2} + q^{2-D}\right) \frac{q^{D} - q^{-D}}{q^2 - q^{-2}}$$

(iii) $b_i = b_i^* =$

$$\frac{q^{D-2} + q^{2-D}}{q^{D-2i} + q^{2i-D}} \frac{q^{2D-2i} - q^{2i-2D}}{q^2 - q^{-2}} \qquad (1 \le i \le D-1)$$

$$b_0 = b_0^* = \left(q^{D-2} + q^{2-D}\right) \frac{q^{D} - q^{-D}}{q^2 - q^{-2}}$$

pf Similar to pf of LEM 23, except use Case I data for θ_i, θ_i^*. $\square$

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<u>Note</u> With reference to Lem 25, assume $D \geq 2$, and observe

$$c_2 - 1 = \frac{q^D + q^{-D}}{q^{D-4} + q^{4-D}}$$

Suppose $c_2 = 1$. Then

$$q^{2D} = -1$$

so $\qquad q^D = i \qquad\qquad i^2 = -1$

Evaluating the data in Lem 25 using $q^D = i$ we find

$$\theta_i = \theta_i^* = q^{2i} + q^{-2i} \qquad (0 \leq i \leq D)$$

$$c_i = c_i^* = 1 \qquad (1 \leq i \leq D-1)$$

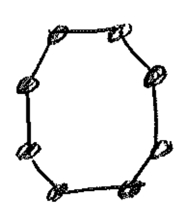
$$c_D = c_D^* = 2$$

$$b_i = b_i^* = 1 \qquad (1 \leq i \leq D-1)$$

$$b_0 = b_0^* = 2$$

Unique solution is ordinary cycle with $2D$ vertices. For example if $D = 4$,

Γ:



LEM 26 Assume Γ is bipartite, and the Q-poly ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite.

Further assume $\beta = -2$.

Then D is even and

(i) $\theta_i = \theta_i^* = (-1)^i (D-2i)$ $\quad (0 \le i \le D)$

(ii) $c_i = c_i^* = i$ $\quad (0 \le i \le D)$

(iii) $b_i = b_i^* = D-i$. $\quad (0 \le i \le D)$

pf Similar to pf of LEM 23, except use Case III data for θ_i and θ_i^* $\square$

Note Unique solution to LEM 26 is hypercube $H(D,2)$ with D even, using the Q-poly ordering of the eigenvalues

$$D,\ 2-D,\ D-4,\ 6-D,\ \ldots$$

(ex show this really is a Q-poly ordering)