

Combinatorics 475

Lecture 22

7.2 Generating Functions

Given a sequence of real numbers

$$\{h_n\}_{n=0}^{\infty}$$

Corresp generating function is

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$x = \text{variable}$

Formal sum — don't consider convergence

To understand h_n sometimes easier to work with $g(x)$

ex

$$h_n = 1$$

h_n

$$g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

ex

$$h_n = \frac{1}{n!}$$

$n = 0, 1, 2, \dots$

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

ex Given integer $m \geq 0$

Consider sequence

$$\binom{m}{0}, \binom{m}{1}, \dots, \binom{m}{m}$$

Find gen function.

Sol. View sequence as ∞ by adding 0 terms

$$\binom{m}{0}, \dots, \binom{m}{m}, 0, 0, \dots$$

$$g(x) = \sum_{n=0}^{\infty} \binom{m}{n} x^n$$

$$= (1+x)^m$$

Ex Fix integer $k \geq 1$

For $n \geq 0$ let

$h_n = \#$ integer solutions to

$$e_1 + e_2 + \dots + e_k = n$$

$$e_1, e_2, \dots, e_k \geq 0$$

Find gen function for $\{h_n\}_{n=0}^{\infty}$

Sol I We saw earlier

$$h_n = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

So

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} h_n x^n \\ &= \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n \end{aligned}$$

$$= (1-x)^{-k}$$

by Ch 5.5

Sol II

Consider

$$(1-x)^{-k}$$

Recall

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$= \sum_{e=0}^{\infty} x^e$$

So

$$(1-x)^{-k} = \frac{1}{1-x} \frac{1}{1-x} \dots \frac{1}{1-x}$$

$$= \left(\sum_{e_1=0}^{\infty} x^{e_1} \right) \left(\sum_{e_2=0}^{\infty} x^{e_2} \right) \dots \left(\sum_{e_k=0}^{\infty} x^{e_k} \right)$$

$$= \sum_{e_1=0}^{\infty} \sum_{e_2=0}^{\infty} \dots \sum_{e_k=0}^{\infty} x^{e_1+e_2+\dots+e_k}$$

$$= \sum_{n=0}^{\infty} \left(\begin{array}{l} \# \text{ of integer solutions to} \\ e_1+e_2+\dots+e_k=n \\ e_1 \geq 0, e_2 \geq 0, \dots, e_k \geq 0 \end{array} \right) x^n$$

$$= \sum_{n=0}^{\infty} h_n x^n$$

$$= g(x)$$

□

Ex Given $n \geq 0$

How many ways to pick n fruits from
an unlimited supply of
apples, oranges, bananas, pears

such that

- $\underbrace{\# \text{ apples}}_{e_1} \text{ is even}$
- $\underbrace{\# \text{ oranges}}_{e_2} \leq 2$
- $\underbrace{\# \text{ bananas}}_{e_3} \text{ divisible by } 3$
- $\underbrace{\# \text{ pears}}_{e_4} \leq 1$

Sol Call this h_n

$h_n =$ # non neg integer solutions to

$$e_1 + e_2 + e_3 + e_4 = n$$

$$e_1 \text{ even}, \quad e_2 \leq 2, \quad 3 | e_3, \quad e_4 \leq 1$$

Gen function

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$$= (1 + x^2 + x^4 + \dots) (1 + x + x^2) (1 + x^3 + x^6 + \dots) (1 + x)$$

$$= \frac{1}{1-x^2} \cdot \frac{1-x^3}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1-x^2}{1-x}$$

$$= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n$$

So $h_n = n+1$

□

Ex Given $n \geq 0$

Find the number of integer solutions to

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n$$

$$e_1 \geq 0, e_2 \geq 0, e_3 \geq 0, e_4 \geq 0$$

Sol Change variables:

$$E_1 = 3e_1$$

$$E_2 = 4e_2$$

$$E_3 = 2e_3$$

$$E_4 = 5e_4$$

* becomes

$$E_1 + E_2 + E_3 + E_4 = n$$

$$E_1, E_2, E_3, E_4 \geq 0$$

$$3|E_1, \quad 4|E_2, \quad 2|E_3, \quad 5|E_4$$

Gen function is

$$g(x) = (1 + x^3 + x^6 + \dots)(1 + x^4 + x^8 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^5 + x^{10} + \dots)$$

$$= \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^5}$$

$$\text{ans} = \text{coeff of } x^n \text{ in } g(x)$$

□

Ex Given integer $n \geq 1$

For a permutation $a_1, a_2, \dots, a_n$ of $\{1, 2, \dots, n\}$

Recall the inversion sequence

$$(b_1, b_2, \dots, b_n)$$

$b_i = \#$ of elements among $a_1, a_2, \dots, a_n$ that are greater than i and to left of i

Recall

$$0 \leq b_1 \leq n-1$$

$$0 \leq b_2 \leq n-2$$

$$0 \leq b_{n-1} \leq 1$$

$$b_n = 0$$

$$\# \text{ inversions of } a_1, a_2, \dots, a_n = b_1 + b_2 + \dots + b_n$$

$$\leq n-1 + n-2 + \dots + 2 + 1 + 0$$

$$= \binom{n}{2}$$

For $0 \leq k \leq \binom{n}{2}$ let

$h_k = \#$ perms of $\{1, 2, \dots, n\}$ that have exactly k inversions

Find gen function for the h_k .

Sol For $0 \leq k \leq (n)$

$h_k = \#$ integer sol to

$$b_1 + b_2 + \dots + b_n = k$$

$$0 \leq b_1 \leq n-1$$

$$0 \leq b_2 \leq n-2$$

$\vdots$

$$0 \leq b_{n-1} \leq 1$$

$$b_n = 0$$

Gen function is $\binom{n}{2}$

$$g(x) = \sum_{k=0} h_k x^k$$

$$= (1+x+x^2+\dots+x^{n-1})(1+x+x^2+\dots+x^{n-2}) \dots (1+x+x^2)(1+x)(1)$$

$$= \frac{1-x^n}{1-x} \frac{1-x^{n-1}}{1-x} \dots$$

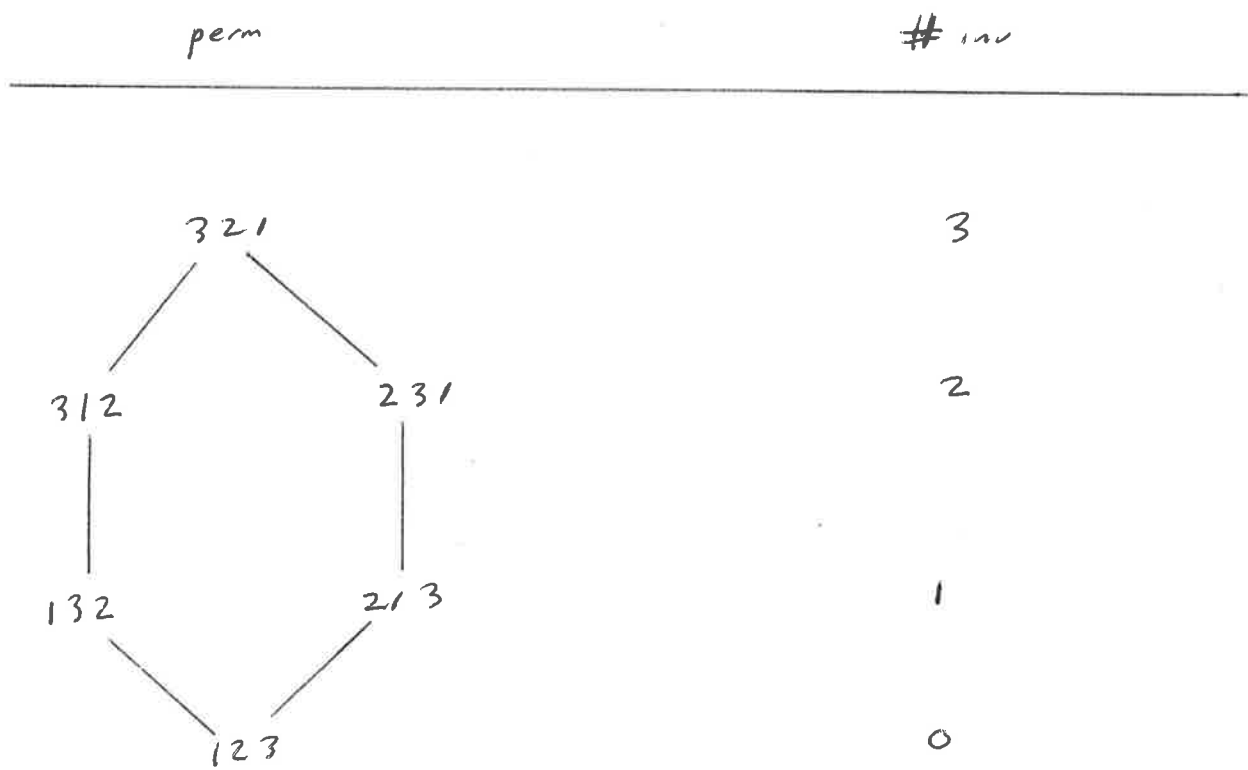
$$\frac{1-x^3}{1-x} \frac{1-x^2}{1-x} \frac{1-x}{1-x}$$

$$= \prod_{j=1}^n \left(\frac{1-x^j}{1-x} \right)$$

To illustrate take $n=3$

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$$g(x) = 1 + 2x + 2x^2 + x^3$$

$$= (1 + x + x^2)(1 + x)$$

