Math 846

Lecture 22

We continue to discuss a DRG $\Gamma = (X, R)$ with diam D that is Q-poly wrt $\{E_i\}_{i=0}^{D}$

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COR 16 Referring to Γ, assume $D \geq 3$ to avoid trivialities.

Let $\beta \in \mathbb{F}$ be from Th 1. Then the eigenvalues $\{\theta_i\}_{i=0}^{D}$ of Γ and the dual eigenvalues $\{\theta_i^*\}_{i=0}^{D}$ of Γ satisfy one of the following forms.

Case I: $\beta \neq \pm 2$

$$\theta_i = a + b q^{2i-D} + c q^{D-2i} \qquad (0 \leq i \leq D)$$
$$\theta_i^* = a^* + b^* q^{2i-D} + c^* q^{D-2i}$$
$$\beta = q^2 + q^{-2}$$

Case II: $\beta = 2$

$$\theta_i = a + bi + ci^2 \qquad (0 \leq i \leq D)$$
$$\theta_i^* = a^* + b^* i + c^* i^2$$

Case III: $\beta = -2$

$$\theta_i = a + b(-1)^i + ci(-1)^i \qquad (0 \leq i \leq D)$$
$$\theta_i^* = a^* + b^*(-1)^i + c^* i(-1)^i$$

Pf By Th 2 and Lem 14. $\square$

Caution: Ref to Cor 16,

possibly some of

$q,\ a,\ b,\ c,\ a^*,\ b^*,\ c^*$

are in $\mathbb{C} \setminus \mathbb{R}$, even though $\theta_i, \theta_i^* \in \mathbb{R}$

for $0 \le i \le D$.

Note 17 Ref to Cor 16, for Case I

the parameters $\beta, \gamma, \gamma^*, \delta, \delta^*$ from Thm 1 are

$$\beta = q^2 + q^{-2}$$

$$\gamma = -a(q-q^{-1})^2$$

$$\gamma^* = -a^*(q-q^{-1})^2$$

$$\delta = -bc(q^2-q^{-2})^2 + a^2(q-q^{-1})^2$$

$$\delta^* = -b^*c^*(q^2-q^{-2})^2 + a^{*2}(q-q^{-1})^2$$

This is checked using Prop 12.

Similar equations hold for Cases II, III

Observe that Γ is bipartite iff $a_i = 0$ for $0 \le i \le D$.

Call our Q-poly ordering $\{E_i\}_{i=0}^D$ <u>dual-bipartite</u> whenever $a_i^* = 0$ for $0 \le i \le D$.

Fix $x \in X$ and write $T = T(x)$ etc.

Define

$$A = \sum_{i=0}^{D} (-1)^i E_i^* \qquad \Delta^* = \sum_{i=0}^{D} (-1)^i E_i$$

LEM 18 (i) Assume Γ is bipartite. Then

$$A \Delta = -\Delta A$$

(ii) Assume the Q-poly ordering $\{E_i\}_{i=0}^D$ is dual bipartite. Then

$$A^* \Delta^* = -\Delta^* A^*$$

pf (i) See pf of Lem 26 in CH1.

(ii) Show

$$A^*\Delta^* + \Delta^* A^* = 0$$

Obs

$$A^*\Delta^* + \Delta^* A^* = I(A^*\Delta^* + \Delta^* A^*) I$$

$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i (A^*\Delta^* + \Delta^* A^*) E_j$$

$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i A^* E_j \left((-1)^j + (-1)^i\right)$$

For $0 \le i, j \le D$

$(-1)^j + (-1)^i = 0$ if i, j have opp parity

$E_i A^* E_j = 0$ if i, j have same parity

So $A^*\Delta^* + \Delta^* A^* = 0$

$\square$

Thm 19 Let the scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ be from Thm 1. Assume $D \geq 2$.

(i) Assume Γ is bipartite. Then

$$0 = A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A$$

and

$$\gamma = 0.$$

(ii) Assume the Q-poly ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite. Then

$$0 = A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*$$

and

$$\gamma^* = 0.$$

pf (i) Define

$$F = A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A$$

show $F = 0$.

Obs

$$F = IFI$$
$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* F E_j^*$$

For $0 \le i, j \le D$ show $E_i^* F E_j^* = 0$

obs $$E_i^* F E_j^* = E_i^* A E_j^* \, p^*(\theta_i^*, \theta_j^*)$$

Case $|i-j| > 1$: $E_i^* A E_j^* = 0$

Case $|i-j| = 1$: $p^*(\theta_i^*, \theta_j^*) = 0$ since the dual eigenvalues are $(\beta, \gamma^*, \delta^*)$-rec

Case $i = j$ $E_i^* A E_j^* = 0$ since $a_i^* = 0$

In all cases $E_i^* F E_j^* = 0$

So $F = 0$

Next show $\gamma = 0$

By TD1 in Thm 1,

$$0 = [A,\ A^2A^* - AA^*A + A^*A^2 - \gamma(AA^*+A^*A) - \delta A^*]$$

In this equation conjugate each side by Δ

and use Lem 18 (i) to get

$$0 = [-A,\ (-A)^2A^* - \beta(-A)A^*(-A) + A^*(-A)^2 - \gamma(-AA^* - A^*A) - \delta A^*]$$

So

$$0 = [A,\ A^2A^* - \beta AA^*A + A^*A^2 + \gamma(AA^*+A^*A) - \delta A^*]$$

Subtracting this from TD1,

$$0 = \gamma[A,\ AA^*+A^*A]$$

$$= \gamma[A^2, A^*]$$

But $[A^2, A^*] \neq 0$ since

$$E_0^*[A^2, A^*]E_2^* = \underbrace{E_0^*A^2E_2^*}_{\neq 0}\ \underbrace{(\theta_2^* - \theta_0^*)}_{\neq 0}$$

$$\neq 0$$

So $\gamma = 0$. (ii) Similar $\square$

Ex Take $\Gamma = H(D,2)$ hypercube.

Recall Γ is bipartite with eigenvalues

$$\theta_i = D - 2i \qquad (0 \le i \le D)$$

Corresp ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial and dual bipartite. The dual eigenvalues are

$$\theta_i^* = D - 2i \qquad (0 \le i \le D)$$

By Prop 12

$$\beta = 2,$$

$$\gamma = 0, \quad \delta = 4, \quad \gamma^* = 0, \quad \delta^* = 4.$$

So by Th 19,

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^*,$$

$$(A^*)^2A - 2A^*AA^* + A(A^*)^2 = 4A$$

(Compare this with Prop 18 in CH1)

Recall our DRG $\Gamma = (X, R)$ with diam $D \geq 1$.

Assume ordering $\{E_i\}_{i=0}^{D}$ is Q-poly.

LEM 20 With above notation, for $0 \leq i \leq D$

(i) $c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta_1 \theta_i^*$

where θ_{-1}^*, θ_{D+1}^* are indets

(ii) $c_i^* \theta_{i-1} + a_i^* \theta_i + b_i^* \theta_{i+1} = \theta_i^* \theta_i$

where θ_{-1}, θ_{D+1} are indets.

pf (i) By Askey-Wilson duality

$$u_i(\theta_1) = u_1^*(\theta_i^*)$$
$$= \theta_i^* / \theta_0^*$$

Result follows from this and the 3-term rec for $\{u_i\}_{i=0}^{D}$

(ii) Similar. $\square$

Setting $i=0$ in LEM 21 (i) and using $b_0 = k = \theta_0$,

$$\frac{\theta_1}{\theta_0} = \frac{\theta_1^*}{\theta_0^*}$$

LEM 21 Assume Γ is bipartite. Then

(i) $$c_i = \frac{\theta_0}{\theta_0^*} \frac{\theta_1^* \theta_i^* - \theta_0^* \theta_{i+1}^*}{\theta_{i-1}^* - \theta_{i+1}^*} \quad (1 \leq i \leq D-1)$$

(ii) $$c_D = \theta_0$$

(iii) $$b_i = \theta_0 - c_i \quad (0 \leq i \leq D)$$

(iv) $$\frac{\theta_{D-1}^*}{\theta_D^*} = \frac{\theta_1^*}{\theta_0^*}$$

pf (i) In L20(i) Eval using $a_i = 0$,
$b_i = k - c_i$, and solve for c_i.

(ii) $c_D = k - b_D = k = \theta_0$

(iii) $k = \theta_0$

(iv) Set $i = D$ in L20(i) $\square$

LEM 22 Assume the Q-polynomial ordering $\{E_i\}_{i=0}^{D}$ is dual bipartite. Then

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(i) $$c_i^* = \frac{\theta_0^*}{\theta_0} \frac{\theta_1\theta_i - \theta_0\theta_{i+1}}{\theta_{i-1} - \theta_{i+1}} \qquad (1 \le i \le D-1)$$

(ii) $$c_D^* = \theta_0^*$$

(iii) $$b_i^* = \theta_0^* - c_i^* \qquad (0 \le i \le D)$$

(iv) $$\frac{\theta_{D-1}}{\theta_0} = \frac{\theta_1}{\theta_0}$$

pf Similar to LEM 21 □