

Combinatorics 475

Lecture 21

7. Recurrence Relations and Generating Functions

7.1 Some number sequences

Consider a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$:

$$a_0, a_1, a_2, \dots$$

Gives a sequence of partial sums $\{s_n\}_{n=0}^{\infty}$

$$s_n = a_0 + a_1 + \dots + a_n$$

$$n = 0, 1, 2, \dots$$

Examples

- Arithmetic sequences

$$\underbrace{a_n - a_{n-1}}_{k} \text{ is indep of } n \text{ for } n \geq 1$$

Here

$$a_n = a_0 + nk$$

$$n = 0, 1, 2, \dots$$

$$a_0 + a_1 + \dots + a_n = (n+1)a_0 + k \binom{n+1}{2}$$

• Geometric sequences

$$\underbrace{a_n / a_{n-1}}_q \text{ indep of } n \text{ for } n \geq 1$$

Here

$$a_n = a_0 q^n$$

$$n = 0, 1, 2, \dots$$

$$a_0 + a_1 + \dots + a_n = a_0(1 + q + q^2 + \dots + q^n)$$

$$= \begin{cases} a_0 \frac{q^{n+1} - 1}{q - 1} \\ a_0(n+1) \end{cases}$$

$$\text{if } q \neq 1$$

$$\text{if } q = 1$$

The Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$

Satisfies

$$f_n = f_{n-1} + f_{n-2}$$

$$n \geq 2$$

$$f_0 = 0, \quad f_1 = 1$$

n	0	1	2	3	4	5	6	7	8	9	10	...
f_n	0	1	1	2	3	5	8	13	21	34	55	...

Find partial sums

$$f_0 + f_1 + \dots + f_n$$

n	0	1	2	3	4	5	6	7	8	...	n
$f_0 + f_1 + \dots + f_n$	0	1	2	4	7	12	20	33	54	...	$f_{n+2} - 1$?

LEM $\forall n \geq 0$

$$f_0 + f_1 + \dots + f_n = f_{n+2} - 1$$

pf By induction on n

$$n=0: \quad \begin{array}{ccc} & ? & \\ f_0 & = & f_2 - 1 \\ \text{"} & & \text{"} \\ 0 & & 1 \end{array} \quad \checkmark$$

$n \geq 1$:

$$\begin{array}{lcl} f_0 + f_1 + \dots + f_{n-1} + f_n & \stackrel{?}{=} & f_{n+2} - 1 \\ \underbrace{\hspace{10em}}_{\text{" ind}} & & \text{"} \\ f_{n+1} & & f_{n+1} + f_n \end{array} \quad \checkmark$$

Next goal: Find f_n in closed form

First ignore initial cond $f_0 = 0, f_1 = 1$

Consider any sequence that satisfies

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2$$

*

Hunt for solutions of form

$$f_n = x^n \quad x \in \mathbb{R}$$

Require

$$x^n = x^{n-1} + x^{n-2} \quad n \geq 2$$

so

$$x^2 = x + 1$$

Find x :

$$x^2 - x - 1 = 0$$

By quad formula

$$x = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Both sequences

$$f_n = \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

$$f_n = \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

satisfy *

Now for any $a, b \in \mathbb{R}$ the sequence

$$f_n = a \left(\frac{1+\sqrt{5}}{2} \right)^n + b \left(\frac{1-\sqrt{5}}{2} \right)^n$$

satisfies *

Now bring in init cond

$$\rightarrow b = -a$$

$$0 = f_0 = a + b$$

$$1 = f_1 = a \frac{1+\sqrt{5}}{2} + b \frac{1-\sqrt{5}}{2}$$

$$= a \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right)$$

$$= a\sqrt{5}$$

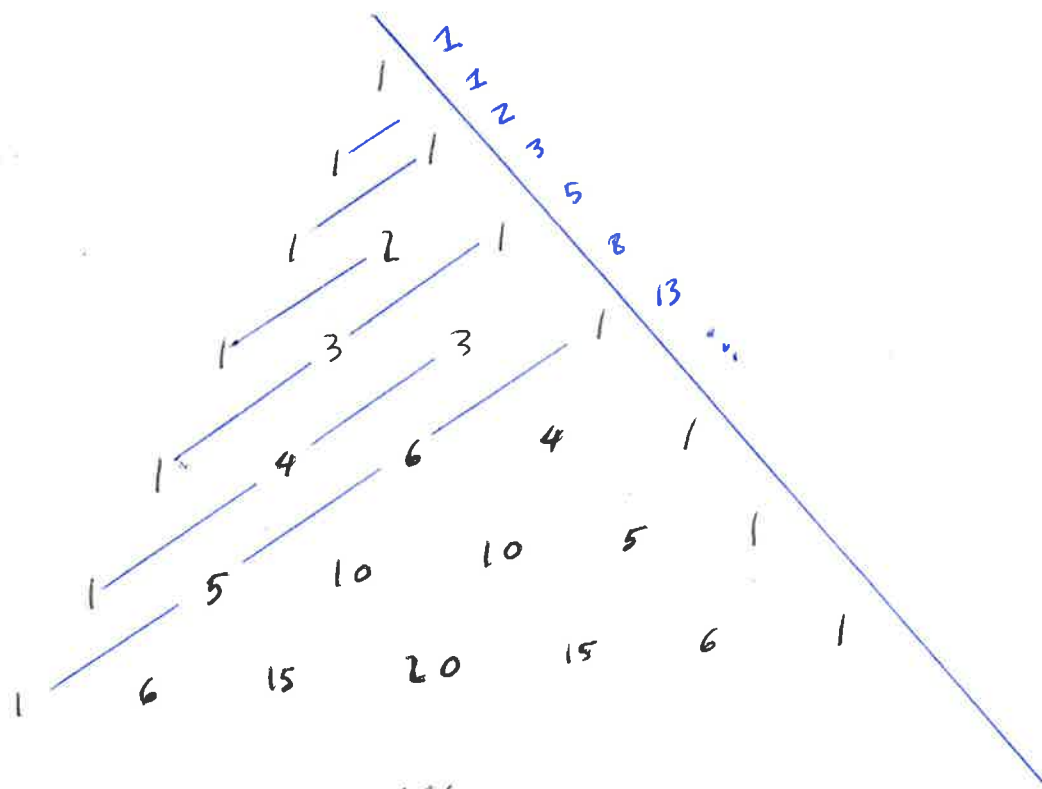
$$a = \frac{1}{\sqrt{5}} \quad b = -\frac{1}{\sqrt{5}}$$

So

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

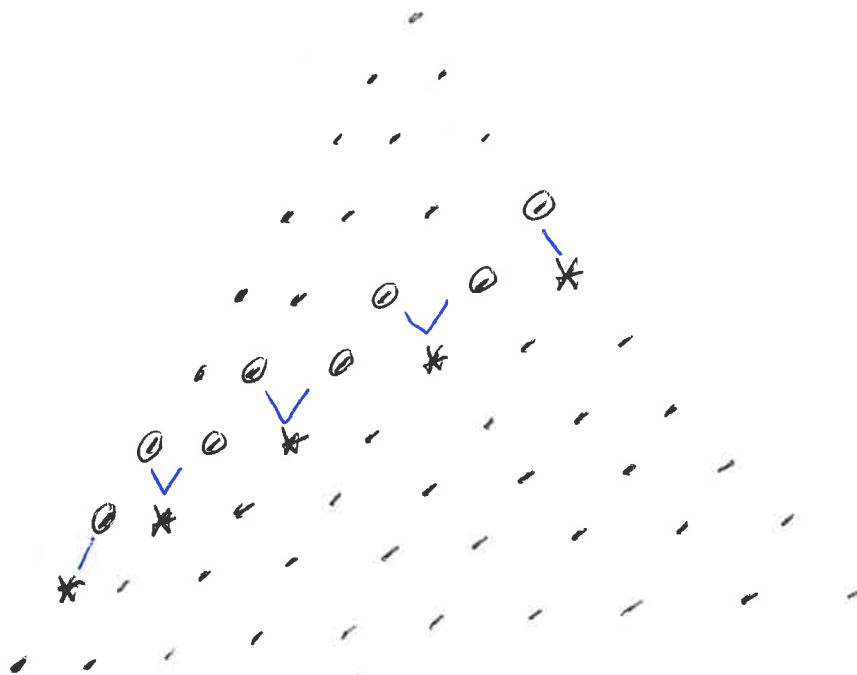
$$n = 0, 1, 2, \dots$$

Recall Pascal's triangle



show $F_n = F_{n-1} + F_{n-2}$

even



show

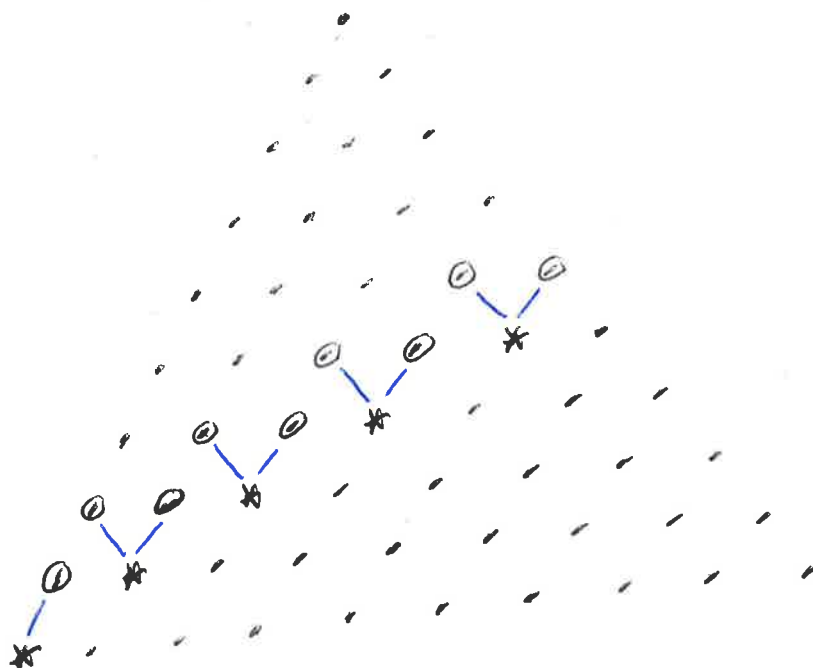
sum of @

=

sum of *

follows from
Pascal's formula

odd



We mention a basic fact about Fibonacci numbers

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LEM For $n \geq 1$

f_n, f_{n+1} are relatively prime

pf Ind on n

$n=1$

$f_1=1$

$f_2=1$ ✓

$n \geq 2$: Given pos integer x such that
 $x \mid f_n$ $x \mid f_{n+1}$

show $x=1$

$$f_{n+1} = f_n + f_{n-1}$$

$x \mid f_{n+1}$ and $x \mid f_n$ so $x \mid f_{n-1}$

x divides $\underbrace{f_n, f_{n-1}}_{\text{rel prime}}$

$x=1$

□

The Fibonacci numbers have the following remarkable properties (ex)

- For integers $r, a \geq 0$,

$$f_r f_a + f_{r+1} f_{a+1} = f_{r+a+1}$$

ex $r=2 \quad a=3$

$$\begin{array}{cc} f_2 & f_3 \\ \parallel & \parallel \\ 1 & 2 \end{array} + \begin{array}{cc} f_3 & f_4 \\ \parallel & \parallel \\ 2 & 3 \end{array} = \begin{array}{c} ? \\ f_6 \\ \parallel \\ 8 \end{array}$$

OK

- $F_n \quad n \geq 1$

f_n divides $f_{2n}, f_{3n}, f_{4n}, \dots$

ex $n=3$

f_3	f_6	f_9	f_{12}
$\parallel$	$\parallel$	$\parallel$	$\parallel$
2	8	34	144
	$\parallel$	$\parallel$	$\parallel$
	2^3	$2 \cdot 17$	$2^4 \cdot 3^2$

- For positive integers m, n that are relatively prime, then f_m, f_n are relatively prime.

ex $m=7$ $n=9$

$$f_7 = 13$$

$$f_9 = 34 = 2 \cdot 17$$

13, 34 rel prime ✓

- For positive integers m, n define

$$d = \text{GCD}(m, n)$$

↑
greatest common divisor

then

$$f_d = \text{GCD}(f_m, f_n)$$

ex

$$m=6$$

$$n=9$$

$$d = \text{GCD}(6, 9) = 3$$

$$f_3 = 2$$

$$f_6 = 8 = 2^3$$

$$f_9 = 34 = 2 \cdot 17$$

$$2 = \text{GCD}(8, 34)$$

✓

