Math 846

Lecture 21

We continue to discuss a DRG $\Gamma=(X,R)$ with diameter $D\geq 1$, that is Q-polynomial with respect to $\{E_i\}_{i=0}^{D}$. Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.

Fix $x\in X$ and write $T=T(x)$ etc.

<u>Proof of Thm 1</u> First assume $D\geq 3$.

By LEM5 (with $R=A^2$ and $S=A$) $\exists$ $Z\in M$ s.t.

$$A^2A^*A - AA^*A^2 = ZA^* - A^*Z \qquad (*)$$

Recall $\{A^i\}_{i=0}^{D}$ is a basis for M, so $\exists$ $p\in\mathbb{F}[\lambda]$ with degree $\leq D$ s.t. $Z=p(A)$. Let $d=\deg(p)$.

We show $d=3$. First suppose $d>3$. Multiply each term in $(*)$ on left by E_d^* and on right by E_0^*. Evaluate using LEM 4 to get

$$0 = \underset{\neq 0}{c}\,(\underset{\neq 0}{\theta_0^*-\theta_d^*})\,\underbrace{E_d^*A^dE_0^*}_{\neq 0 \text{ by LEM3}} \qquad c = \text{leading coef of } p$$

Contradiction. Next suppose $d<3$. Multiply each term in $(*)$ on left by E_3^* and on right by E_0^*. Evaluate using LEM 4 to get

$$\underset{\neq 0}{(\theta^*-\theta_2^*)}\,\underbrace{E_3^*A^3E_0^*}_{\neq 0 \text{ by LEM3}} = 0$$

for a contradiction.

We have shown $d=3$. Abbrev $\beta = c^{-1} - 1$

Divide both sides of $(*)$ by c to find $\exists\, \gamma, \delta \in \mathbb{F}$ s.t.

$$(\beta+1)(A^2A^*A - AA^*A^2) = A^3A^* - A^*A^3$$

$$-\gamma\,(A^2A^* - A^*A^2) - \delta\,(AA^* - A^*A)$$

Rearranging terms, we get TD1. To get TD2, pick i $(2 \le i \le D-1)$. Multiply each term in TD1 on the left by E^*_{i-2} and on the right by E^*_{i+1}.

Simplify using LEM 4 to get

$$0 = \underbrace{E^*_{i-2} A^3 E^*_{i+1}}_{\neq 0} \Big(\underbrace{\theta^*_{i-2} - (\beta+1)\,\theta^*_{i-1} + (\beta+1)\,\theta^*_i - \theta^*_{i+1}}_{\text{must be } 0} \Big)$$

So $\{\theta^*_i\}_{i=0}^{D}$ is β-recurrent.

So $\exists\, \gamma^* \in \mathbb{F}$ s.t. $\{\theta^*_i\}_{i=0}^{D}$ is (β, γ^*)-rec

So $\exists\ \delta^* \in \mathbb{F}$ st $\{\theta_i^*\}_{i=0}^D$ is $(\beta, \gamma^*, \delta^*)$-rec.

We have

$$P^*(\theta_{i-1}^*, \theta_i^*) = 0 \qquad (1 \le i \le D)$$

Now $\beta, \gamma^*, \delta^*$ satisfy TD2 by LEM 7*

We are done for $D \ge 3$.

Next assume $D < 3$. Let $\beta \in \mathbb{F}$ (arbitrary)

If $D = 2$ define $\gamma = \theta_0 - \beta\theta_1 + \theta_2$

and if $D \le 1$ let $\gamma \in \mathbb{F}$ (arb).

By constr $\{\theta_i\}_{i=0}^D$ is (β, γ)-rec.

So $\exists\ \delta \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^D$ is (β, γ, δ)-rec.

We have $\quad p(\theta_{i-1}, \theta_i) = 0 \qquad (1 \le i \le D)$

Now β, γ, δ satisfy TD1 by Lem 7.

Interchanging the roles of A, A^* in the above argument, we find $\exists\ \gamma^*, \delta^* \in \mathbb{F}$ st

$\beta, \gamma^*, \delta^*$ satisfy TD2.

$\square$

Prop 12 Given $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ that satisfy TD1, TD2.

(i) the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are both equal to $\beta + 1$ for $2 \le i \le D-1$.

(ii) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}$ $(1 \le i \le D-1)$

(iii) $\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ $(1 \le i \le D-1)$

(iv) $\delta = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i)$ $(1 \le i \le D)$

(v) $\delta^* = \theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1} + \theta^*_i)$ $(1 \le i \le D)$

pf (iv) By TD1 and Lem 7

(v) By TD2 and Lem 7*

(ii) By LEM 11 and (iv)

(iii) By LEM 11 and (v)

(i) $\{\theta_i\}_{i=0}^D$ is (β, γ)-rec by (ii) so $\{\theta_i\}_{i=0}^D$ is β-rec.

Similarly $\{\theta^*_i\}_{i=0}^D$ is β-rec. The result follows. $\square$

Cor 13 The scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ in THM 1 are unique if $D \geq 3$.

pf By Prop 12. $\square$

Thm 2 is immediate from Prop 12 (i).

Our next goal is to solve the recurrence in Thm 2 to get the eigenvalues and dual eigenvalues of Γ in closed form. Lec 21 6

LEM 14 Given a sequence of scalars $\{\sigma_i\}_{i=0}^{D}$ in $\mathbb{C}$, and given $\beta \in \mathbb{C}$. Then $\{\sigma_i\}_{i=0}^{D}$ is β-recurrent iff $\exists\, a, b, c \in \mathbb{C}$ such that

Case $\beta \neq \pm 2$

$$\sigma_i = a + b q^i + c q^{-i} \qquad (0 \le i \le D)$$

where $\beta = q + q^{-1}$

Case $\beta = 2$

$$\sigma_i = a + bi + ci^2 \qquad (0 \le i \le D)$$

Case $\beta = -2$

$$\sigma_i = a + b(-1)^i + ci(-1)^i \qquad (0 \le i \le D)$$

pf (For case $\beta \neq \pm 2$) Assume $\rho \geq 3$; else trivial.

Let L denote the set of all vectors $(\sigma_0, \sigma_1, \ldots, \sigma_\rho) \in \mathbb{C}^{\rho+1}$

that are β-rec, ie

$$\sigma_{i-2} - (\beta+1)\sigma_{i-1} + (\beta+1)\sigma_i - \sigma_{i+1} = 0 \qquad (2 \leq i \leq \rho-1) \qquad (*)$$

L is a subspace of $\mathbb{C}^{\rho+1}$.

In $(*)$, $\sigma_0, \sigma_1, \sigma_2$ are free and $\sigma_3, \sigma_4, \ldots, \sigma_\rho$ are

determined by $\sigma_0, \sigma_1, \sigma_2$. So

$$\dim L = 3$$

Pick $0 \neq q \in \mathbb{C}$ st

$$\beta = q + q^{-1}$$

obs $q \neq 1$, $q \neq -1$

One checks the three vectors

$$(1, 1, 1, \ldots, 1) \qquad (1, q, q^2, \ldots, q^\rho), \qquad (1, q^{-1}, q^{-2}, \ldots, q^{-\rho})$$

are in L and lin indep. So they form a basis for L.

Result follows. $\square$

Note 15 Ref to LEM 14 for $\beta \neq \mp 2$ Lec 21 8

sometimes we replace q by q^2 and adjust a, b, c

to write

$$\sigma_i = a + b q^{2i-D} + c q^{D-2i} \qquad (0 \le i \le D)$$