

Combinatorics 475

Lecture 20

Problem Given integers $n, k \geq 1$

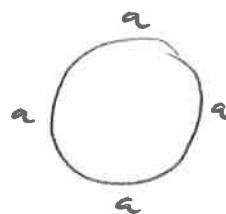
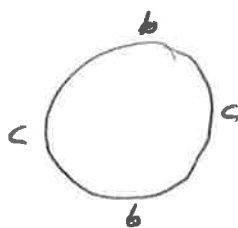
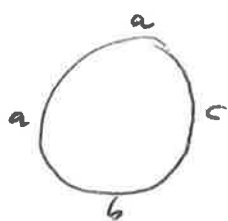
Find the number of circular n -permutations of

$$\{ \infty = a_1, \infty = a_2, \dots, \infty = a_k \}$$

the multiset

$$\{ \infty = a_1, \infty = a_2, \dots, \infty = a_k \} \quad *$$

Ex $n=4$ $k=3$ $a_1=a$ $a_2=b$ $a_3=c$



period:

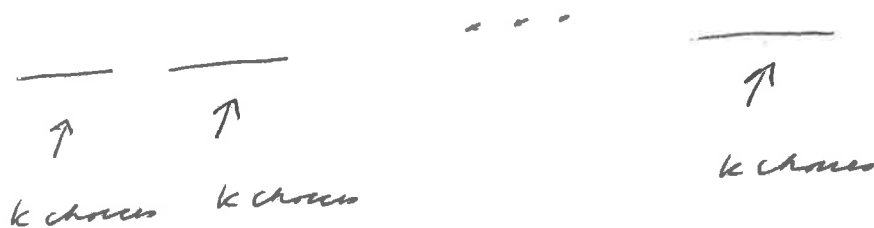
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1

First consider (ordinary) n -perms of *

To construct one fill in blanks:



n -perms of * is k^n

Now consider function

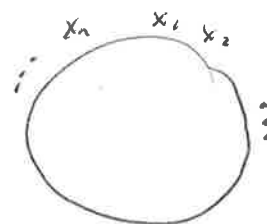
n -perms of X



circular n -perms
of X

Wrap:

$x_1, x_2, \dots, x_n$



wrap around circle

ex

Wrap:

a c b a



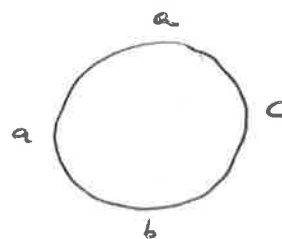
c b a a



b a a c



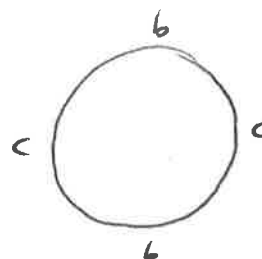
a a c b



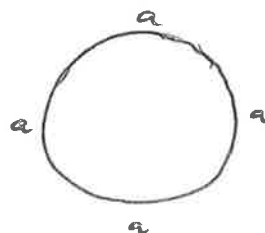
b c b c



c b c b



a a a a



Given circular n -perm θ of $\mathbb{K}$

$$\left| \left\{ n\text{-perms of } \mathbb{K} \text{ that wrap around to } \theta \right\} \right| = \text{Period}(\theta)$$

For each pos integer d that divides n

let $C(n, d) = \# \text{ circular } n\text{-perms of } \mathbb{K} \text{ that have period exactly } d$

$$\text{So } C(n) = \sum_{d|n} C(n, d) \quad (1)$$

For $d|n$ find $C(n, d)$

First, Construct n -perms of $\mathbb{K}$ that yield a circular n -perm of $\mathbb{K}$ with period divisible by d :

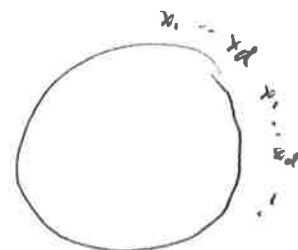
For any d -perm $x_1 x_2 \dots x_d$ of $\mathbb{K}$

Consider n -perm

$$\underbrace{x_1 x_2 \dots x_d}_{n/d \text{ copies}} \quad \underbrace{x_1 x_2 \dots x_d}_{n/d \text{ copies}} \quad \dots \quad \underbrace{x_1 x_2 \dots x_d}_{n/d \text{ copies}}$$

n/d copies

wrap
→



period divides d

this yields

$$k^d = \sum_{e|d} e C(n, e)$$

(2)

[one eq for each divisor d of n]

Apply Mobius inversion to (2):

$$\forall d|n,$$

$$dC(n, d) = \sum_{e|d} \mu(d/e) k^e$$

So

$$C(n, d) = \frac{1}{d} \sum_{e|d} \mu(d/e) k^e$$

Now by (1)

$$C(n) = \sum_{d|n} C(n, d)$$

$$= \sum_{d|n} \frac{1}{d} \sum_{e|d} \mu(d/e) k^e$$

$$= \frac{1}{n} \sum_{d|n} \frac{n}{d} \sum_{e|d} \mu(d/e) k^e$$

$$= \frac{1}{n} \sum_{e|n} k^e \sum_{d|n} \frac{n}{d} \mu(d/e)$$

[change order of summation]

change var
 $d = n/e$
 $N = n/e$

$$\sum_{d|n} \frac{n}{d} \mu(d/e)$$

$$=$$

$$\phi(N)$$

$$= \phi(n/e)$$

$$= \frac{1}{n} \sum_{e|n} \phi(n/e) k^e$$

We have shown

Lec 20

5

Then For $n, k \geq 1$

The number of circular n -permutations of the multiset

$$\{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_k\}$$

is

$$\frac{1}{n} \sum_{e|n} \phi(e) k^e$$

where

ϕ = Euler ϕ -function

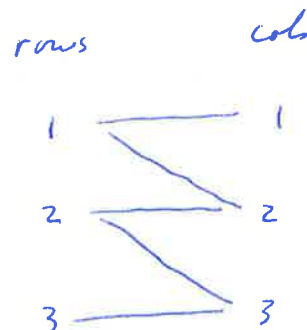
□

Given $n \times n$ matrix A with entries 0 or 1

Represent A by diagram:

$n=3$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



Observe

ways to put n nonattacking rooks on 1-entries of A

= # n -matchings of diagram

Find this number.

Let R = set of rows of $A = \{1, 2, \dots, n\}$

C = set of cols of $A = \{1, 2, \dots, n\}$

View n -matching of diagram as a bijection

$$f: R \rightarrow C$$

such that

$$x \mapsto f(x) \quad \forall x \in R$$

Use incl excl to find #

Case A has all entries 1

$$\text{So } x \mapsto y \quad \forall x \in R \quad \forall y \in C$$

Compute # of n -matchings 2 ways.

Calc I

$$\# \text{ n-matchings} = n!$$

Calc II

Let $S =$ set of all functions $R \rightarrow C$

$$|S| = n^n$$

$\forall i \in S_n$ define

$$A_i = \{ f \in S \mid i \notin \underset{\substack{\uparrow \\ \text{image of } f}}{\text{Im}(f)} \}$$

obs

$$\# \text{ n-matchings} = | \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_n} |$$

$\forall A \subseteq \{1, 2, \dots, n\}$ define

$$A_\Delta = \bigcap_{i \in \Delta} A_i$$

$$A_\Delta = \{ f \in S \mid \text{Im}(f) \subseteq \{1, 2, \dots, n\} \setminus \Delta \}$$

$$|A_\Delta| = (n - |\Delta|)^n$$

By incl/excl

$$|\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n| = \sum_{A \subseteq \{1, 2, \dots, n\}} |A_A| (-1)^{|A|}$$

$$= \sum_{k=0}^n \binom{n}{k} (n-k)^n (-1)^k$$

Comparing calc I, II we get

$$n! = \sum_{k=0}^n \binom{n}{k} (n-k)^n (-1)^k$$

$$n = 0, 1, 2, \dots$$

Ex Find the number of perms $a_1 a_2 a_3 a_4 a_5 a_6$

of $\{1, 2, 3, 4, 5, 6\}$ such that

$a_1, a_2 \notin \{1, 2\}$ $a_3, a_4 \notin \{3, 4\}$ $a_5, a_6 \notin \{5, 6\}$

Sol. View as rook placement problem with forbidden positions

	x	x				
x	x					
			x	x		
			x	x		
					x	x
					x	x



For $0 \leq i \leq 6$ find r_i

Helps to view chessboard as defining a graph diagram



rows

cols

$\overset{r}{\curvearrowright} \underset{c}{\curvearrowright}$ means X appears in row r , col c

$r_i = \#$ of matchings of above graph using exactly i edges
"i-matching"

subproblem

For $i=0,1,2$ find # i -matchings for



i	0	1	2
# i -matchings	1	4	2

For convenience define "matching polynomial" in var x :

$$1 + 4x + 2x^2$$

For $i=0,1,2$ coef of x^i = # i -matchings of $\square$

Matching poly for $\square \square \square = (1 + 4x + 2x^2)^3$

$$= 1 + 12x + 54x^2 + 112x^3 + 108x^4 + 48x^5 + 8x^6$$

So for $\boxtimes \boxtimes \boxtimes$

i	0	1	2	3	4	5	6
r_i	1	12	54	112	108	48	8

So

ans = # ways to place 6 nonattacking rooks on chessboard with forb pos *

$$= \sum_{i=0}^6 r_i (6-i)! (-1)^i$$

$$= 6! - 12 \cdot 5! + 54 \cdot 4! - 112 \cdot 3! + 108 \cdot 2! - 48 \cdot 1! + 8 \cdot 0!$$

□

